The non-Gaussian Renormalization Group fixed point of a three-dimensional φ^4 theory

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Based on joint works with

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2^{nd} order phase transitions and universality

Goal: quantitatively understand the behavior of a statmech model at a 2^{nd} order phase transition, e.g.

- compute critical exponents
- prove their universality
- compute scaling limit of correlations and prove their conformal invariance

To clarify what I mean, consider the FM Ising model on $\mathbb{Z}^d$ with $d \geq 2$:

$$H(\sigma) = -\frac{1}{2} \sum_{x,y \in \mathbb{Z}^d} J(x-y)\sigma_x \sigma_y - h \sum_{x \in \mathbb{Z}^d} \sigma_x$$

with $J(x) \geq 0$ and tempered:

$$|J(x)| \leq \frac{C}{|x|^{d+s}}$$

for all $x \neq 0$ and some $C, s > 0$.

The ferromagnetic Ising model, 2

Known facts (consider for simplicity $J(x)$ of finite range):

1. For $h \neq 0$ the model has a unique infinite volume Gibbs measure and

$$\left| \langle \sigma_x \sigma_y \rangle_{\beta,h} - \langle \sigma_x \rangle_{\beta,h} \langle \sigma_y \rangle_{\beta,h} \right| \leq C e^{-\kappa |x-y|}$$

for some $C, \kappa > 0$ (and similarly for higher order correlations).

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2. If $h = 0$, there exists $\beta_c > 0$ such that:

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2. If $h = 0$, there exists $\beta_c > 0$ such that:

1. for $\beta \leq \beta_c$ the model has a unique infinite volume Gibbs measure
2. for $\beta > \beta_c$ the model has multiple infinite volume Gibbs measures, in particular

$$\langle \sigma_x \rangle^{+}_{\beta,0} = -\langle \sigma_x \rangle^{-}_{\beta,0} > 0$$

The ferromagnetic Ising model, 3

3. The phase transition is 2^{nd} order: $\langle \sigma_x \rangle^{+}_{\beta,0}$ is continuous in β, but the specific heat (second order derivative of the pressure w.r.t. β) diverges logarithmically at β_c.

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4. Correlations decay exponentially at all $\beta \neq \beta_c$:

$$\left| \langle \sigma_x \sigma_y \rangle^+_{\beta,0} - \langle \sigma_x \rangle^+_{\beta,0} \langle \sigma_y \rangle^+_{\beta,0} \right| \leq Ce^{-\kappa|x-y|}$$

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for some $C, \kappa > 0$ (similarly for higher order correl.).
5. Expected: at $\beta = \beta_c$ correlations decay polynomially:
$$\langle \sigma_x \sigma_y \rangle_{\beta_c,0} \sim \frac{C_0}{|x-y|^{d-2+\eta}},$$
for some $C_0 > 0$ and $\eta \geq 0$, as $|x-y| \to \infty$.

The critical exponent η

The critical exponent $\eta = \eta(d) \geq 0$ in $d \geq 2$ is:

$$\eta(2) = 1/4$$
$$\eta(3) = 0.03629761\cdots$$
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Note: besides depending upon d and on the short-range nature of the FM coupling, η is expected to be independent of the specific choice of $J(x)$ and, more generally, of the specific microscopic model under consideration, provided it describes a scalar spin displaying $\mathbb{Z}_2$ symmetry.

Other critical exponents

Similar expectations for other critical exponents:
α, b, γ, δ, ν, defined as follows:

$$\begin{aligned}
\frac{\partial^2}{\partial\beta^2}\psi(\beta,0) &\sim c_1(\beta_c-\beta)^{-\alpha} && \text{as } \beta\to(\beta_c)^- \\
\langle\sigma_x\rangle^+_{\beta,0} &\sim c_2(\beta-\beta_c)^{b} && \text{as } \beta\to(\beta_c)^+ \\
\frac{\partial}{\partial h}\langle\sigma_x\rangle_{\beta,h}\Big|_{h=0} &\sim c_3(\beta_c-\beta)^{-\gamma} && \text{as } \beta\to(\beta_c)^- \\
\langle\sigma_x\rangle_{\beta_c,h} &\sim c_4\, h^{1/\delta} && \text{as } h\to 0^+ \\
\xi(\beta) &\sim c_5(\beta_c-\beta)^{-\nu} && \text{as } \beta\to(\beta_c)^-
\end{aligned}$$

[Here $\psi(\beta,h)$ is the pressure, and $\xi(\beta)$ is the correlation length, s.t., for $\beta<\beta_c$, $\langle\sigma_x\sigma_y\rangle_{\beta,0}\simeq e^{-|x-y|/\xi(\beta)}$ as $|x-y|\to\infty$.]

Dependence of critical exponents upon d

	$d = 2$	$d = 3$	$d \geq 4$
α	0	0.11008708	0
b	$1/8$	0.32641871	$1/2$
γ	$7/4$	1.23707551	1
δ	15	4.78984254	3
ν	1	0.62997097	$1/2$
η	$1/4$	0.03629761	0

The 2D exponents are those of the nearest-neighbor Ising model.

The 3D exponents are those computed by conformal bootstrap.

The exponents in $d \geq 4$ are those of the Gaussian 'mean-field' model.

Universality. Ising vs 'soft spin' ϕ^4_d model

The Ising critical exponents are also expected to be the same as those of the following 'soft spin' model:

$$Z_{\beta,\lambda,h} = \int \mathcal{D}\phi \, e^{-\beta \mathcal{H}_{\lambda,h}(\phi)}$$

with

$$\mathcal{H}_{\lambda,h}(\phi) = -\tfrac{1}{2} \iint \phi(x)\phi(y) J(x-y)\, dx\, dy + \lambda \int (\phi^2(x) - 1)^2 \, dx - h \int \phi(x)\, dx$$

where $J(x) \geq 0$ is rapidly decaying and $\lambda > 0$.

[Ultraviolet and infrared cutoffs are implicit in the definition above.]

Universality and Renormalization Group, 1

Robustness of critical exp. known as universality.
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Roughly, the RG ideas are the following:

1. we can coarse grain the system by tiling $\mathbb{Z}^d$ with boxes of side $\gamma > 1$, integrate out local spin fluctuations around mean values of spin in the boxes, rescale distances by $1/\gamma$, thus ending up with new effective model on $\mathbb{Z}^d$, which retains informations about how original system behaved on scales larger than γ, in particular it has the same large distance critical exponents.

Universality and Renormalization Group, 2

2. If we keep iterating the previous averaging+ +rescaling procedure, at criticality we get an infinite trajectory of effective models, equivalent as far as critical exponents are concerned (if the system is off-critical, the trajectory 'stops' – i.e., the model trivializes – once we reach the scale of the correlation length).

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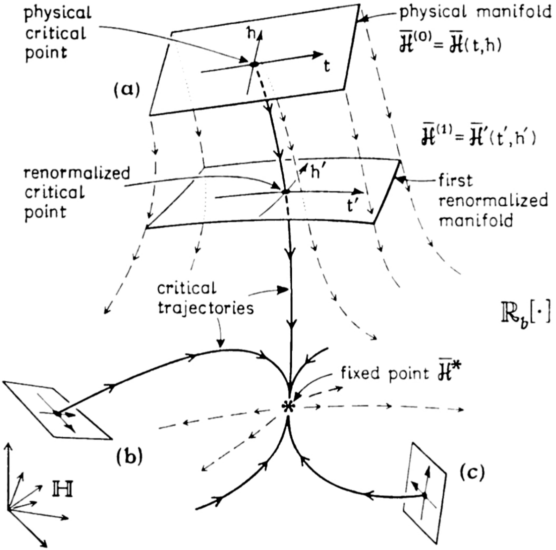
3. At criticality, such infinite trajectory reaches a Fixed Point, which is a Euclidean Field Theory describing the scaling limit of correlations. (By construction, these limiting correlations are scale invariant.)

Pictorial representation of RG trajectory



From soft-spin to ϕ^4_d model, 1

Let us describe the Wilsonian RG procedure more precisely. The Hamiltonian of the soft-spin model, at $h = 0$ and small momenta reads:

$$\mathcal{H}(\phi) = -\tfrac{1}{2}\int |\hat{\phi}(k)|^2 \hat{J}(k)\, dk + \lambda \int (\phi^2(x) - 1)^2\, dx$$

$$\simeq -\tfrac{1}{2}\int |\hat{\phi}(k)|^2 (\hat{J}(0) - \alpha_0 k^2)\, dk + \lambda \int (\phi^2(x) - 1)^2\, dx$$

$$\equiv \tfrac{\alpha_0}{2}\int \left(|\nabla\phi(x)|^2 + \nu_0 \phi^2(x) + \tfrac{2\lambda}{\alpha_0}\phi^4(x)\right) dx + (\mathrm{const.})$$

Under this approximation, after rescaling of ϕ by $\sqrt{\beta\alpha_0}$, the partition function $Z = \int \mathcal{D}\phi e^{-\beta\mathcal{H}(\phi)}$ can be rewritten as:

From soft-spin to ϕ_d^4 model, 2

$$Z \propto \int P_{\nu_0}(\mathcal{D}\phi)\, e^{-\lambda_0 \int \phi^4(x)\, dx}$$

where $P_{\nu_0}(\mathcal{D}\phi)$ is Gaussian measure with covariance

$$\begin{aligned} g_{\nu_0}(x-y) &:= \int P_{\nu_0}(\mathcal{D}\phi)\phi(x)\phi(y) \\ &= \int \frac{dk}{(2\pi)^d} \frac{e^{-ik(x-y)}}{k^2+\nu_0} \chi_{\leq 0}(k) \end{aligned}$$

[Here $\chi_{\leq 0}(k)$ is smooth characteristic funct^n of $|k| \leq 1$.]

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If $\nu_0 \ge 0$, asymptotically as $|x-y| \to \infty$, one has:

$$g_{\nu_0}(x-y) \sim \frac{C_0}{|x-y|^{d-2}} e^{-\sqrt{\nu_0}|x-y|}.$$

The ϕ^4_d model

$$Z_{\lambda_0,\nu_0} := \int P_{\nu_0}(\mathcal{D}\phi) e^{-\lambda_0 \int \phi^4(x)\, dx}$$

is an effective model, parametrized by $\lambda_0 > 0$ and ν_0, known as the (infrared) ϕ^4_d model. For λ_0 small, it can be regarded as a perturbation of a Gaussian measure.

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$$\langle \phi(x)\phi(y) \rangle_{\lambda_0,\nu_c} := \frac{1}{Z_{\lambda_0,\nu_c}} \int P_{\nu_c}(\mathcal{D}\phi) e^{-\lambda_0 \int \phi^4(x)\, dx} \phi(x)\phi(y)$$

decays polynomially at large distances.

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Conjecturally, decay exponent of $\langle \phi(x)\phi(y) \rangle_{\lambda_0,\nu_c}$ is $d - 2 + \eta$, same as Ising in d dimensions.

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Wilsonian idea: rewrite $\chi_{\le 0}(k) = f_0(k) + \chi_{\le -1}(k)$ where $\chi_{\le -1}(k) := \chi_{\le 0}(\gamma k)$ and decompose accordingly $g_{\nu_0} = g^{(0)} + g^{(\le -1)}$, where now $g^{(0)}$ decays (stretched) exponentially at large distances, uniformly in ν_0.

Integration of the fluctuation field

We write $\phi = \phi^{(0)} + \varphi$, with $\phi^{(0)}$ distributed accordingly to Gaussian measure P_0 with covariance $g^{(0)}$, and φ distributed accordingly to Gaussian measure $P_{\leq -1}$ with covariance $g^{(\leq -1)}$.

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We use the addition principle for Gaussian integrals:

$$\int P_{\nu_0}(\mathcal{D}\phi) e^{-\mathcal{V}^{(0)}(\phi)} = \int P_{\leq -1}(\mathcal{D}\varphi) \int P_0(\mathcal{D}\phi^{(0)}) e^{-\mathcal{V}^{(0)}(\phi^{(0)}+\varphi)}$$

$$= e^{F_{-1}} \int P_{\leq -1}(\mathcal{D}\varphi) e^{-V^{(-1)}(\varphi)}$$

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Due to exp. decay of $g^{(0)}$,

$$F_{-1} + V^{(-1)}(\varphi) = \log \int P_0(\mathcal{D}\phi^{(0)}) e^{-\mathcal{V}^{(0)}(\phi^{(0)}+\varphi)}$$

can be computed by a convergent cluster expansion.

Extracting the local quadratic terms

With the purpose in mind of setting the resulting expression in a form as close as possible to the original one, we now extract from $V^{(-1)}(\varphi)$ two local quadratic terms, with 0 and 2 derivatives:

$$V^{(-1)}(\varphi) = \int \left(\delta_{-1}\phi^2(x) + \alpha_{-1}|\nabla\varphi(x)|^2\right) dx + \widetilde{V}^{(-1)}(\phi)$$

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[In order to do this, if $\int |\hat{\varphi}(k)|^2 \hat{W}_2^{(-1)}(k)\, dk$ is the quadratic part of $V^{(-1)}(\varphi)$, we Taylor expand $\hat{W}_2^{(-1)}(k) = \hat{W}_2^{(-1)}(0) + \frac{k^2}{2}\partial^2 \hat{W}_2^{(-1)}(0)+$ $+\hat{R}_2^{(-1)}(k)$ and anti-trasform the first two terms.]

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We then combine the local quadratic terms with the reference Gaussian measure, thus 'dressing' it as:

Dressing the Gaussian measure

$$\int P_{\nu_0}(\mathcal{D}\phi)e^{-\mathcal{V}^{(0)}(\phi)} = e^{\mathcal{F}_{-1}} \int \widetilde{P}_{\leq -1}(\mathcal{D}\varphi)e^{-\widetilde{V}^{(-1)}(\varphi)},$$

where $\widetilde{P}_{\leq -1}$ is Gaussian measure with covariance

$$\widetilde{g}^{(\leq -1)}(x-y) = \frac{1}{\zeta_{-1}} \int \frac{dk}{(2\pi)^d} \frac{e^{-ik(x-y)}}{k^2 + \widetilde{\nu}_{-1}} \chi_{\leq -1}(k)$$

where $\zeta_{-1} = 1 + \alpha_{-1}$ and $\zeta_{-1}\widetilde{\nu}_{-1} = \nu_0 + \delta_{-1}$.

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Recall: $\chi_{\leq -1}(k) = \chi_{\leq 0}(\gamma k) \Rightarrow$ rescale $k = \gamma^{-1}k'$ in the integral expression of $\widetilde{g}^{(\leq -1)}$, thus getting

$$\widetilde{g}^{(\leq -1)}(x-y) = \frac{\gamma^{-(d-2)}}{\zeta_{-1}}\int \frac{dk'}{(2\pi)^d}\frac{e^{-ik'\gamma^{-1}(x-y)}}{|k'|^2+\nu_{-1}}\chi_{\leq 0}(k')$$

where $\nu_{-1} = \gamma^2\widetilde{\nu}_{-1}$.

Rescaling. The Wilsonian RG map

Therefore, if we rescale φ as follows:

$$\varphi(x) = \gamma^{-(d-2)/2}\zeta_{-1}^{-1/2}\phi(\gamma^{-1}x) \equiv \Theta_{-1}\phi(x)$$

where Θ is the rescaling operator, we find:

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$$\mathcal{V}^{(-1)} = R_{-1}\mathcal{V}^{(0)}$$

which consists in:

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which consists in:

- 'integrating out' the fluctuation field
- 'dressing' the Gaussian measure
- 'rescaling' the infrared field.

RG trajectory and fixed point

We now iterate the procedure, so that, for all $h \leq 0$:

$$\int P_{\nu_0}(\mathcal{D}\phi)e^{-\mathcal{V}^{(0)}(\phi)} = e^{\mathcal{F}_h}\int P_{\nu_h}(\mathcal{D}\phi)e^{-\mathcal{V}^{(h)}(\phi)},$$

where $\mathcal{V}^{(h)} = R_h\mathcal{V}^{(h+1)}$, and

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Correspondingly, we expect that

$$\lim_{h\to-\infty} \mathcal{V}^{(h)} = \mathcal{V}^*,$$

the Fixed Point (FP) potential, which defines a scale-invariant Euclidean Field Theory.

Remarks

1. The FP potential satisfies $\mathcal{V}^* = R^* \mathcal{V}^*$ where $R^* \equiv R_{-\infty}$ involves the rescaling $\Theta^* \equiv \Theta_{-\infty}$ s.t.

$$\Theta^* \phi(x) := \gamma^{-(d-2)/2} (\zeta^*)^{-1/2} \phi(\gamma^{-1} x) \equiv \gamma^{-(d-2+\eta)/2} \phi(\gamma^{-1} x),$$

where $\zeta^* := \lim_{h \to -\infty} \zeta_h$ and $\eta := \log_\gamma \zeta^*$.

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2. Universality classes are associated with the basins of attractions under the RG flow of different FP potentials. This motivates the interest in constructing solutions to the RG FP equation and in studying their stability.

3. $\mathcal{V}^* = 0$ is always a trivial solution to the RG FP equation, referred to as the Gaussian (or trivial) FP.

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This suggests that, for $d > 4$, $\lim_{h\to-\infty} \lambda_h = 0$, i.e., the FP potential has no local quartic term. Similar considerations hold for higher order terms $\Rightarrow \mathcal{V}^*$ is trivial. [These considerations are at the basis of the proof that in $d > 4$, in the vicinity of the origin, there is no RG FP other than $\mathcal{V}^* = 0$.]

Remarks

5. In $d = 4$, an explicit computation of ρ_h as a function of λ_h at lowest non-trivial order shows that $\lambda_h \to 0$ as $h \to -\infty$ qualitatively as follows:

$$\lambda_h \simeq \frac{\lambda_0}{1 + \frac{9}{2\pi^2}\lambda_0 |h|}.$$

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These results rigorously substantiate prediction that in $d \geq 4$ the critical exponents of FM Ising are the same as the Gaussian, mean-field, model.

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The non-trivial FP of 3D Ising has only been constructed under uncontrolled approximations, such as the hierarchical approximation.

Remarks

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Truncating at third order, and setting $\epsilon = 1$ gives $\eta(3) \simeq 0.03721\cdots$, in remarkable agreement with the 'correct' value 0.03629761.

Remarks

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In this case it is possible to rigorously prove existence of a non-trivial FP, with qualitative features similar to those of the Wilson-Fisher FP.

Another route: conformal invariance

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Before getting there, let us comment about a possible strategy used to compute the critical exponents of the *real* 3D Ising model, via CFT.

Idea: under additional locality assumptions (existence of a local energy-momentum tensor), the scale-invariant EFT associated with $\mathcal{V}^*$ is actually conformal invariant

OPE and the conformal bootstrap program

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This approach led to full classification of 2D CFTs and to a way to numerically estimate at a higher and higher precision the critical exponents of 3D CFTs (conformal bootstrap).

What does it mean to classify and solve CFTs?

Euclidean CFT on $\mathbb{R}^d$: collection of conformally invariant correlation functions $\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle$ of a complete basis of scaling operators $\mathcal{O}_i(x)$ (such as the scaling limits of the spin and the energy operators in the case of the FM Ising model, or $\phi, \phi^2, \phi\nabla\phi, \ldots$ in the case of the ϕ^4_d model), whose correlations, in the simplest case (scalar *primary* fields), transform as follows:

$$\langle \mathcal{O}_{i_1}(x_1') \cdots \mathcal{O}_{i_n}(x_n') \rangle = \Big(\prod_{j=1}^{n} \Omega(x_j)^{-\Delta_{i_j}} \Big) \langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle$$

where $\Omega(x) = |\partial x' / \partial x|^{1/d}$ and Δ_i is the scaling dimension of $\mathcal{O}_i$.

Structure of correlation functions

Conformal invariance fixes the 2- and 3-point fcts:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}$$

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{\lambda_{ijk}}{|x_{12}|^{\Delta_{ijk}}|x_{13}|^{\Delta_{ikj}}|x_{23}|^{\Delta_{jki}}}$$

for some real λ_{ijk}, where $x_{ij} = x_i - x_j$ and $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$.

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Note: conformal invariance alone is not sufficient for fixing the structure of higher point correlations

Operator Product Expansion

OPE: within correlation functions we can replace

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k \lambda_{ijk} \hat{f}_{ijk}(x, y, z, \partial_z)\mathcal{O}_k(z)$$

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In $d \geq 3$: **conformal bootstrap** aims at computing spectrum and structure constants from applying OPE to 4-point function in different ways.

Axiomatic vs constructive CFT, 1

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Proving that this is the case is a very important but very hard problem, unsolved but for Gaussian FPs and few other integrable cases, such as 2D Luttinger model (Benfatto, Gallavotti, Mastropietro, Falco, ...)

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The only 3D model for which a non-Gaussian FP has been rigorously constructed is the ϕ_3^4 model with fractional Laplacian mentioned above (Brydges, Mitter, Scoppola, Abdesselam, Slade, ...).

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It is a possible testing ground for the conjecture that its fixed point is a CFT. In principle it should be possible to construct microscopically the scaling operators and prove that their correlations satisfy the conformal bootstrap axioms.

Long range Ising model FP, 1

FM long-range Ising on $\mathbb{Z}^d$:

$$H(\sigma) = -J \sum_{x \neq y} \frac{\sigma_x \sigma_y}{|x-y|^{d+s}}, \quad d = 2, 3, \quad s > 0.$$

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- $s > s^*$: same as n.n. Ising in dim. d.

Long range Ising model FP, 2

For $s = d/2 + \epsilon$ with $0 < \epsilon \ll 1$, non-trivial FP accessible by rigorous methods. Proceeding as discussed above for short range Ising model, the problem consists in the computation of

$$\int P_{\leq 0}(d\phi) e^{-\nu_0 \int \phi^2 - \lambda_0 \int \phi^4}$$

and of corresponding correlation functions, where $P_{\leq 0}(d\phi)$ is Gaussian measure with propagator

$$g^{(\leq 0)}(x-y) = \int P_{\leq 0}(d\phi)\phi(x)\phi(y)$$
$$:= \int \frac{dk}{(2\pi)^d} \frac{\chi_{\leq 0}(k)}{|k|^{d/2+\epsilon}} e^{-ik(x-y)}$$

Scalar vs fermionic long-range ϕ^4_d

Rigorous RG construction of this long-range, fractional, ϕ^4_d performed in $d \leq 3$ by Brydges, Mitter, Scoppola, Abdesselam, Slade, but analysis complicated by **large field** problem.

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Simpler, but still non-trivial, setting: 'fermionic' counterpart of long-range ϕ^4_d (Gawedzki-Kupiainen 1985).

Fermionic long-range ϕ^4_d: main results

Our results, in short:

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Many things still to be done, but setting amenable to study validity of CFT axioms and related issues.

Next time we will define the fermionic long-range ϕ^4_d precisely, discuss the nature of its RG fixed point and state our main results.

Thank you!