Ornstein-Zernike asymptotics in Statistical Mechanics

Yvan Velenik
Université de Genève

based on joint works with M. Campanino and D. Ioffe

Contents

Random walks in Physics

Random walks are often used in Physics as simple effective models for more complicated objects.

Three examples to be discussed:

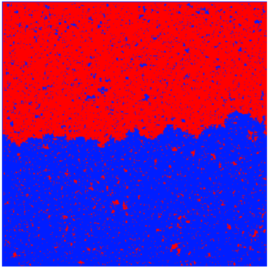
- Interfaces in 2d
- Subcritical clusters
- Stretched polymers

Random walks in Physics

Random walks are often used in Physics as simple effective models for more complicated objects.

Three examples to be discussed:

- Interfaces in 2d
- Subcritical clusters
- Stretched polymers

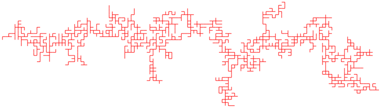


Random walks in Physics

Random walks are often used in Physics as simple effective models for more complicated objects.

Three examples to be discussed:

- Interfaces in 2d
- Subcritical clusters
- Stretched polymers

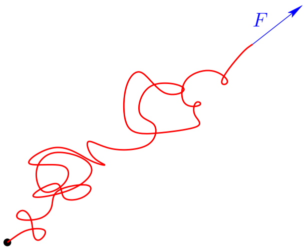


Random walks in Physics

Random walks are often used in Physics as simple effective models for more complicated objects.

Three examples to be discussed:

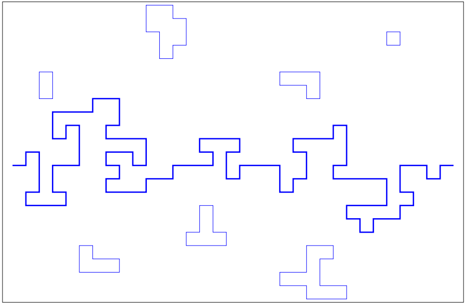
- Interfaces in 2d
- Subcritical clusters
- Stretched polymers



Example: 1D Interface

Problem: Analysis of the statistical properties of an interface separating two equilibrium phases.

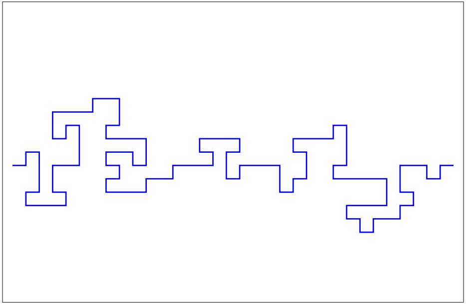
Complicated geometry!
Is it possible to analyse instead a simpler model?



Example: 1D Interface

Problem: Analysis of the statistical properties of an interface separating two equilibrium phases.

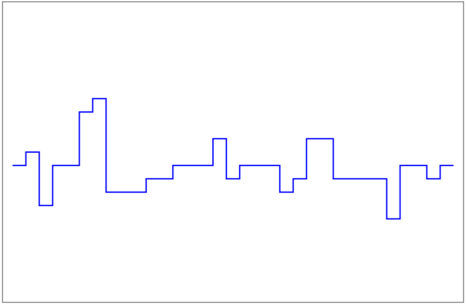
Complicated geometry!
Is it possible to analyse instead a simpler model?



Example: 1D Interface

Problem: Analysis of the statistical properties of an interface separating two equilibrium phases.

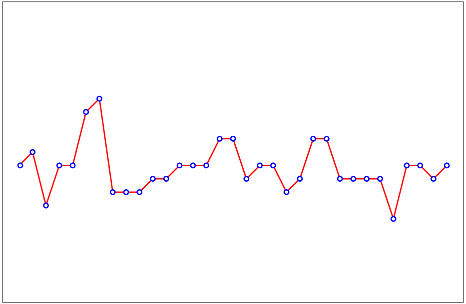
Complicated geometry!
Is it possible to analyse instead a simpler model?



Example: 1D Interface

Problem: Analysis of the statistical properties of an interface separating two equilibrium phases.

Complicated geometry!
Is it possible to analyse instead a simpler model?



Heuristic arguments

- Structure should be simple at microscales large compared to the correlation length.
- Large scale fluctuations should display universal (Brownian) asymptotics.

Contents

Ornstein-Zernike theory

Proposed in 1914 by L. S. Ornstein and F. Zernike.

Aim: Determine the (large distance) behaviour of density-density correlations in simple fluid, away from criticality.

Ornstein-Zernike theory

OZ equation:

$$G(x_1 - x_2) = C(x_1 - x_2) + \rho \int C(x_1 - x_3) G(x_3 - x_2)\, \mathrm{d}x_3$$

- $g(x)$: pair correlation function
- $G(x) = g(x) - 1$: net correlation function
- $C(x)$: direct correlation function

Ornstein-Zernike theory

Fourier transform:

$$\widehat{G}(k) = \frac{\widehat{C}(k)}{1 - \rho\widehat{C}(k)}$$

Ornstein-Zernike theory

Fourier transform:

$$\widehat{G}(k) = \frac{\widehat{C}(k)}{1 - \rho\widehat{C}(k)}$$

Crucial assumption: Separation of masses

C has smaller range than G (faster exp. decay)

Ornstein-Zernike theory

Fourier transform:

$$\widehat{G}(k) = \frac{\widehat{C}(k)}{1 - \rho\widehat{C}(k)}$$

Possible to expand:

$$\widehat{C}(k) \approx \widehat{C}(0) - R^2|k|^2$$

Ornstein-Zernike theory

Fourier transform:

$$\widehat{G}(k) \approx \frac{\widehat{C}(0)}{\rho R^2(\kappa^2 + |k|^2)}$$

Possible to expand:

$$\widehat{C}(k) \approx \widehat{C}(0) - R^2|k|^2$$

Ornstein-Zernike theory

This yields

OZ asymptotics

$$G(x) \approx \frac{A}{|x|^{(d-1)/2}}\, e^{-\kappa|x|}$$

(valid as $|x| \to \infty$ with κ fixed)

Contents

Goal: Rigorous understanding of both (closely related) problems from first principles. Already well understood 30 years ago, in **perturbative** regime:

- OZ: [Abraham, Kunz '77], [Paes-Leme '78], [Bricmont, Fröhlich '85], etc.
- Scaling of 2D Ising interface: [Gallavotti '72], [Higuchi '79], [Bricmont, Fröhlich, Pfister '81], etc.

Goal: Rigorous understanding of both (closely related) problems from first principles. Already well understood 30 years ago, in **perturbative** regime:

- OZ: [Abraham, Kunz '77], [Paes-Leme '78], [Bricmont, Fröhlich '85], etc.
- Scaling of 2D Ising interface: [Gallavotti '72], [Higuchi '79], [Bricmont, Fröhlich, Pfister '81], etc.

Nonperturbative results for very simple models: SAW [Chayes, Chayes '86, Ioffe '98], percolation [Campanino, Chayes, Chayes '91]. Very model-specific approaches.

Goal: Rigorous understanding of both (closely related) problems from first principles. Already well understood 30 years ago, in **perturbative** regime:

- OZ: [Abraham, Kunz '77], [Paes-Leme '78], [Bricmont, Fröhlich '85], etc.
- Scaling of 2D Ising interface: [Gallavotti '72], [Higuchi '79], [Bricmont, Fröhlich, Pfister '81], etc.

Nonperturbative results for very simple models: SAW [Chayes, Chayes '86, Ioffe '98], percolation [Campanino, Chayes, Chayes '91]. Very model-specific approaches.

Robust nonperturbative approach: percolation [Campanino, Ioffe '02], Ising model [Campanino, Ioffe, V '03], random-cluster model [Campanino, Ioffe, V '08], selfinteracting polymers [Ioffe, V '08].

EXAMPLE #1

Interfaces in 2D systems

q-states Potts model

- $\Lambda \Subset \mathbb{Z}^2$
- $\sigma_i \in \{1, \ldots, q\}$, $\forall i \in \Lambda$
- $\beta \geq 0$

$$\pi_\Lambda^{\beta,q}(\sigma) \propto \exp\Bigl(\beta \sum_{i \sim j} \delta_{\sigma_i,\sigma_j}\Bigr)$$

In particular, for $q = 2$: **Ising model**

q-states Potts model

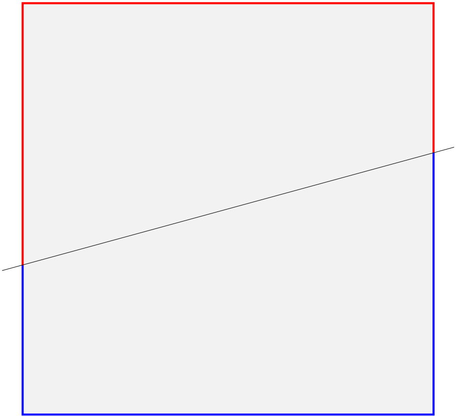
There exists $\beta_c \in (0, \infty)$ such that

- For all $\beta < \beta_c$: unique equilibrium phase
- For all $\beta > \beta_c$: q different equilibrium phases

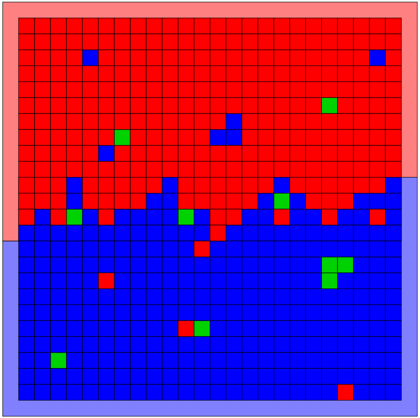
Assumption:

$$\boxed{\beta > \beta_c}$$

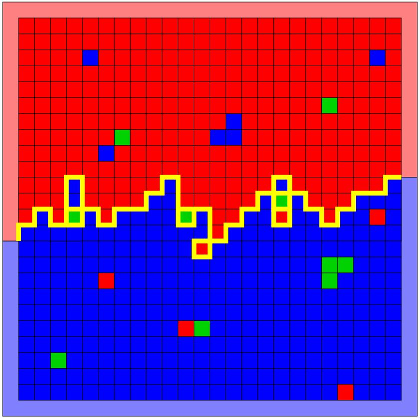
Dobrushin boundary condition



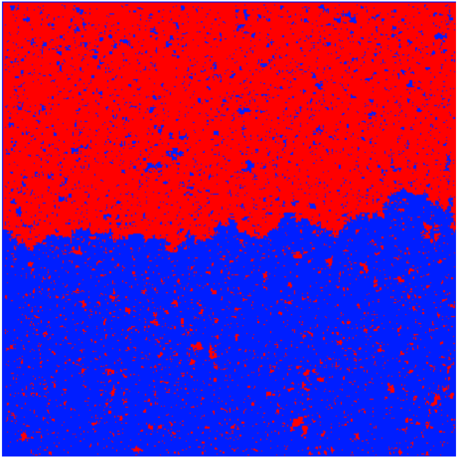
Dobrushin boundary condition



Dobrushin boundary condition

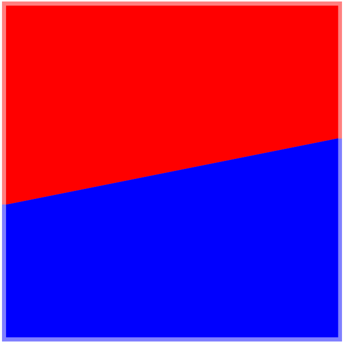


Dobrushin boundary condition



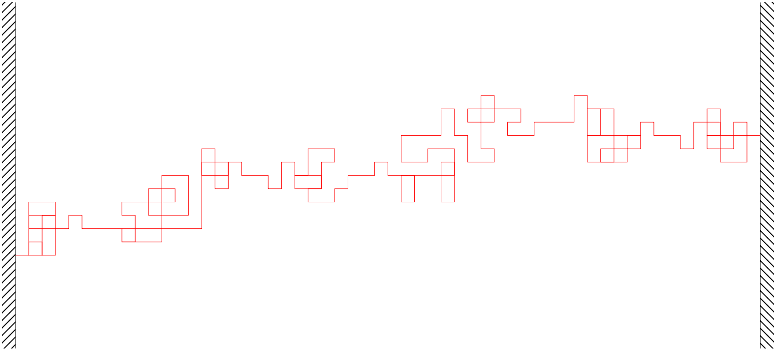
Geometry of the interface

Macro-scale: not very interesting



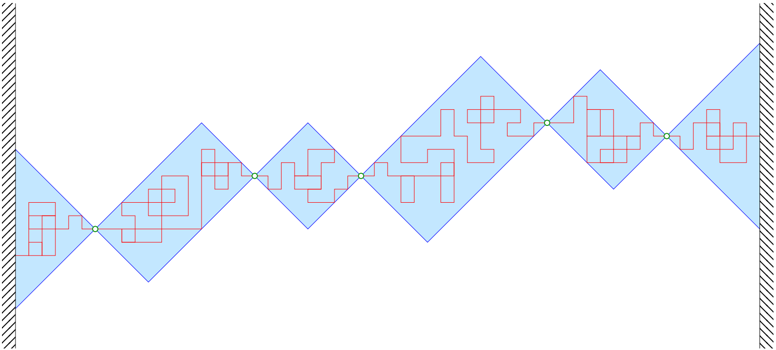
Geometry of the interface

Micro-scale: effective random walk picture



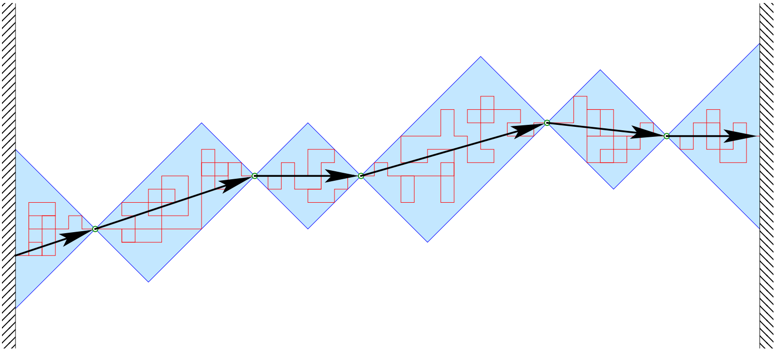
Geometry of the interface

Micro-scale: effective random walk picture



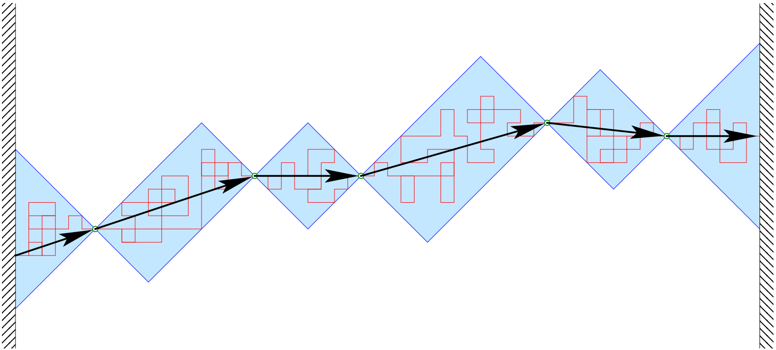
Geometry of the interface

Micro-scale: effective random walk picture



Geometry of the interface

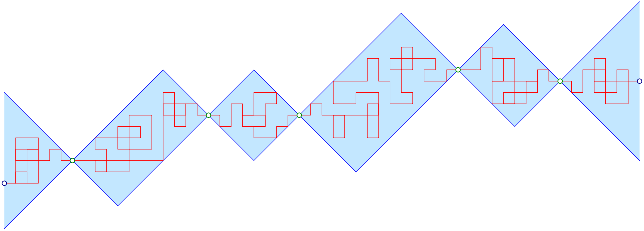
Micro-scale: effective random walk picture



(⚠ Steps are not independent!)

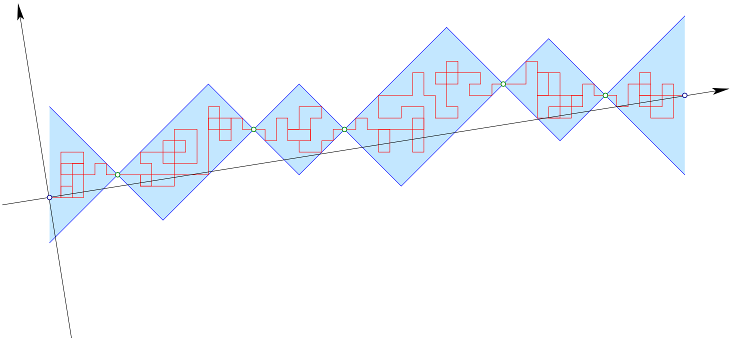
Geometry of the interface

Meso-scale: interface fluctuations



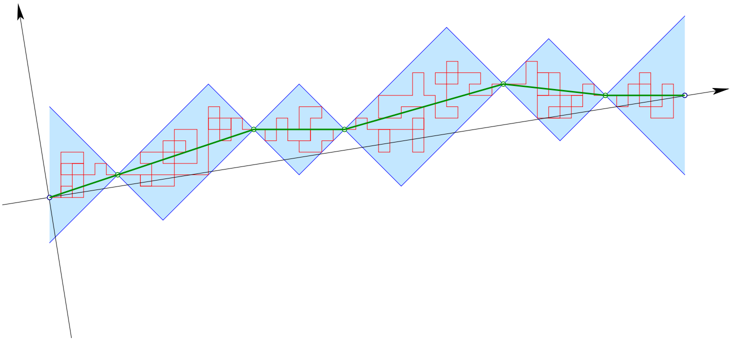
Geometry of the interface

Meso-scale: interface fluctuations



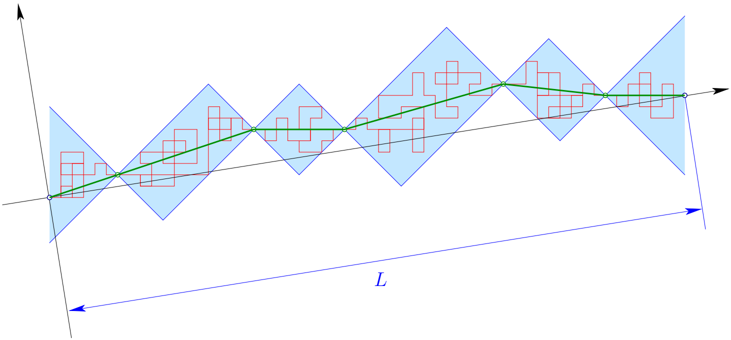
Geometry of the interface

Meso-scale: interface fluctuations



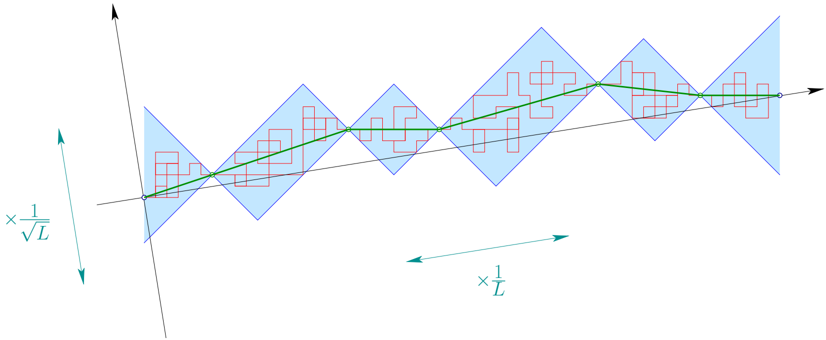
Geometry of the interface

Meso-scale: interface fluctuations



Geometry of the interface

Meso-scale: interface fluctuations



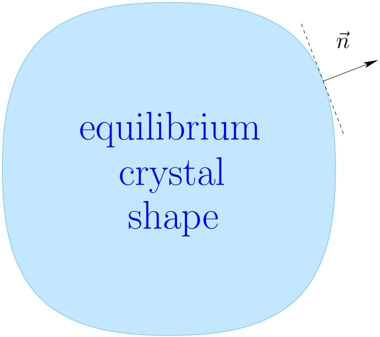
Geometry of the interface

Result

Convergence to $\sqrt{\chi_{\beta,q}(\vec{n})}\ B$

B: standard Brownian bridge on $[0,1]$
$\chi_{\beta,q}(\vec{n})$: curvature of the (Wulff) equilibrium crystal shape
$\vec{n}$: normal to the (macroscopic) interface

Geometry of the interface



Equilibrium crystal shape: (deterministic) shape of a macroscopic droplet of one equilibrium phase immersed inside another.

Geometry of the interface

Moreover, we can deduce that the equilibrium crystal shape possesses

- an analytic boundary,
- a uniformly positive curvature.

Geometry of the interface

Moreover, we can deduce that the equilibrium crystal shape possesses

- an analytic boundary
- a uniformly positive curvature

In particular: no roughening in 2D Potts models.

EXAMPLE #2

Large clusters in subcritical percolative systems

FK percolation

(a.k.a. *random cluster model*)

- Introduced by Fortuin and Kasteleyn in 1972.
- 2 parameters: $p \in [0,1]$, $q \in \mathbb{R}^+$.
- Reduces to Bernoulli percolation when $q = 1$, and to the q-states Potts model for $q = 2, 3, 4, \ldots$

FK percolation

(a.k.a. *random cluster model*)

- Introduced by Fortuin and Kasteleyn in 1972.
- 2 parameters: $p \in [0,1]$, $q \in \mathbb{R}^+$.
- Reduces to Bernoulli percolation when $q = 1$, and to the q-states Potts model for $q = 2, 3, 4, \ldots$

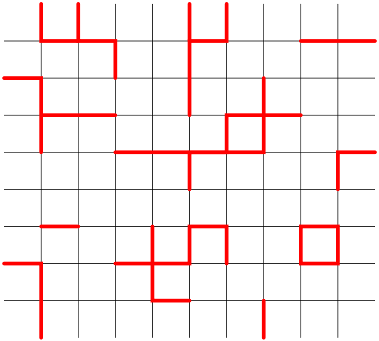
In the sequel: $q \geq 1$.

FK percolation

$\Lambda \Subset \mathbb{E}^d$

FK percolation

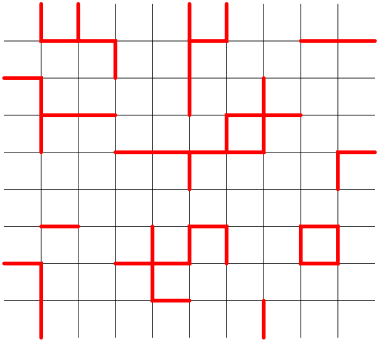
$\Lambda \Subset \mathbb{E}^d$
$\omega \in \{0,1\}^\Lambda$



FK percolation

$\Lambda \Subset \mathbb{E}^d$
$\omega \in \{0,1\}^\Lambda$

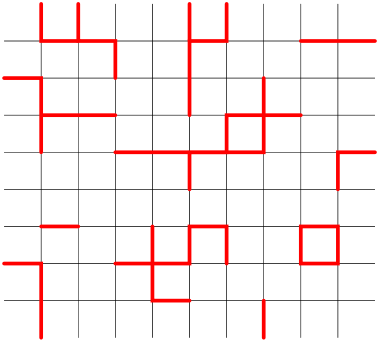
$p \in [0,1]$
$q \in \mathbb{R}, q \geq 1$
$N(\omega) =$ number of clusters



FK percolation

$\Lambda \Subset \mathbb{E}^d$
$\omega \in \{0,1\}^\Lambda$

$p \in [0,1]$
$q \in \mathbb{R}, q \geq 1$
$N(\omega) =$ number of clusters



$$\mathbb{P}_\Lambda^{p,q}(\omega) \propto \prod_b p^{\omega_b}(1-p)^{1-\omega_b}\, q^{N(\omega)}$$

FK percolation

Infinite volume probability measure: $\mathbb{P}^{p,q}$

FK percolation

Infinite volume probability measure: $\mathbb{P}^{p,q}$

There exists $p_c(q,d) > 0$ such that

$$\begin{aligned} \mathbb{P}^{p,q}(0 \leftrightarrow \infty) &= 0 \qquad \forall p < p_c(q,d) \\ \mathbb{P}^{p,q}(0 \leftrightarrow \infty) &> 0 \qquad \forall p > p_c(q,d) \end{aligned}$$

FK percolation

Infinite volume probability measure: $\mathbb{P}^{p,q}$

There exists $p_c(q,d) > 0$ such that

$$\mathbb{P}^{p,q}(0 \leftrightarrow \infty) = 0 \qquad \forall p < p_c(q,d)$$
$$\mathbb{P}^{p,q}(0 \leftrightarrow \infty) > 0 \qquad \forall p > p_c(q,d)$$

In the sequel, we shall always assume that $p < p_c(q,d)$.
(In particular: unique infinite volume measure.)

Connectivity function

Basic quantity: connectivity function

$$\mathbb{P}^{p,q}(0 \leftrightarrow x)$$

Reduces to Potts model 2-point correlation function when $q = 2, 3, 4, \ldots$

Connectivity function

Assumption: $p < p_c(q)$ (slightly cheating here)

In particular, there exists $\xi_{p,q}(\vec{n}) > 0$ such that

$$\mathbb{P}^{p,q}(0 \leftrightarrow x) \leq e^{-\xi_{p,q}(\vec{n}_x)|x|} \qquad (\vec{n}_x = x/|x|)$$

OZ behaviour

Result

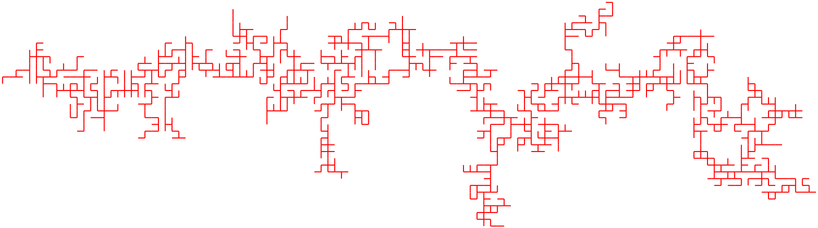
$$\mathbb{P}^{p,q}(0 \leftrightarrow x) = \frac{\Psi_{p,q}(\vec{n}_x)}{|x|^{(d-1)/2}} \, \mathrm{e}^{-\xi_{p,q}(\vec{n}_x)\,|x|} \, (1 + o(1))$$

uniformly as $|x| \to \infty$. The functions $\Psi_{p,q}$ and $\xi_{p,q}$ are positive, analytic functions on $\mathbb{S}^{d-1}$.

In particular: OZ behaviour for Potts 2-point correlation functions

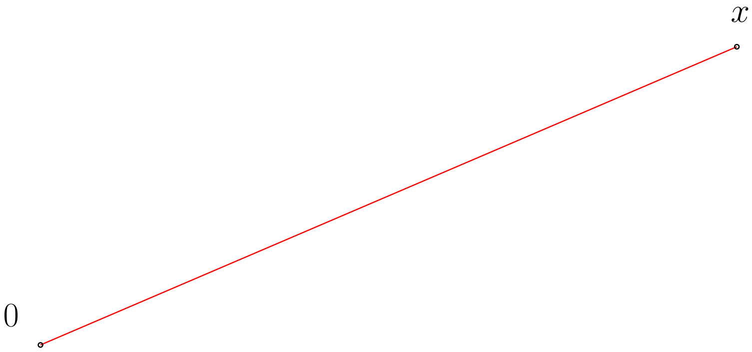
Geometry

Typical shape of large clusters under $\mathbb{P}^{p,q}(\,\cdot\,|\,0 \leftrightarrow x)$ $(|x| \gg 1)$?



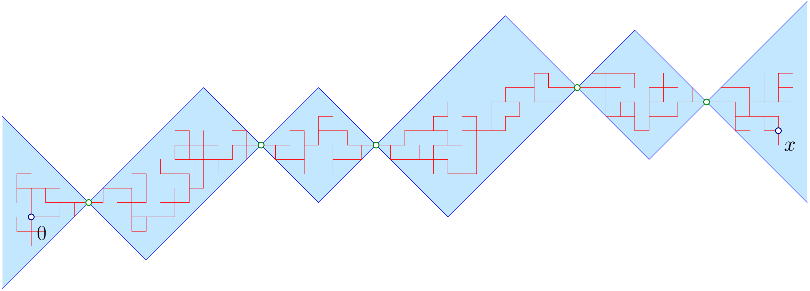
Geometry

Macro-scale: not very interesting



Geometry

Micro-scale: effective random walk picture



(⚠ Again, steps are not independent, except for $q = 1$)

Geometry

Meso-scale: fluctuations

Result

After similar scaling as before, convergence to

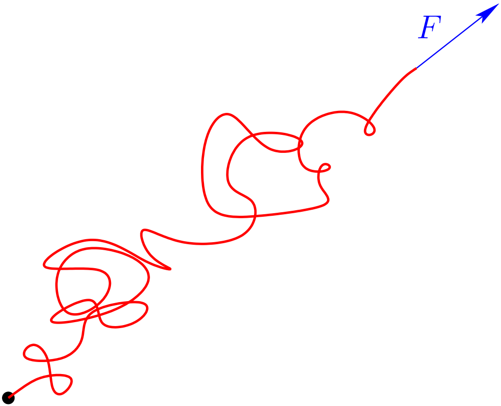
$$\left(\sqrt{\chi^1_{\beta,q}(x)}\; B^1, \dots, \sqrt{\chi^{d-1}_{\beta,q}(x)}\; B^{d-1}\right)$$

B^k: indep. standard Brownian bridges on $[0,1]$
$\chi^k_{\beta,q}(x)$: principal curvatures of "Wulff shape" associated to $\xi_{p,q}$

EXAMPLE #3

Selfinteracting polymers

Selfinteracting polymers



Selfinteracting polymers

$$\mathbb{P}_n^F(\gamma) \propto e^{-\Phi(\gamma)+\langle F,\gamma(n)\rangle}$$

- Polymer chain: $\gamma = (\gamma(0), \dots, \gamma(n))$

  (nearest-neighbour path on $\mathbb{Z}^d$)
- Force applied to free end: F
- Local times: $\ell_x(\gamma) = \sum_{k=0}^n \mathbf{1}_{\{\gamma(k)=x\}}$

  (also possible with edges)
- Potential: $\Phi(\gamma) = \sum_{x\in\mathbb{Z}^d} \phi(\ell_x(\gamma))$

  $\phi \geq 0$, nondecreasing, $\phi(0) = 0$

Selfinteracting polymers

Possible assumptions on the interaction:

- **Repulsive**: $\phi(n+m) \geq \phi(n) + \phi(m)$

Selfinteracting polymers

Possible assumptions on the interaction:

- **Repulsive**: $\phi(n+m) \geq \phi(n) + \phi(m)$
- **Attractive**: $\phi(n+m) \leq \phi(n) + \phi(m)$

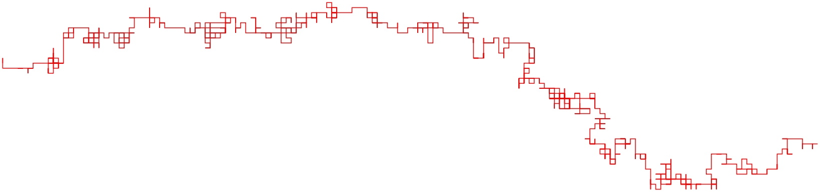
Selfinteracting polymers

Possible assumptions on the interaction:

- **Repulsive**: $\phi(n+m) \geq \phi(n) + \phi(m)$
- **Attractive**: $\phi(n+m) \leq \phi(n) + \phi(m)$
- **Small perturbations** of the pure cases, e.g.,
  - Mixed interactions (e.g., strong attr.+weak rep.)
  - Selfinteracting (e.g., reinforced) RW with drift

Geometry

Typical shape of long polymers under $\mathbb{P}_n^F$?



Phase transition

Attractive case: Transition between a collapsed phase and a stretched phase.

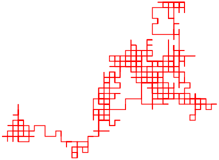
Phase transition

Attractive case: Transition between a collapsed phase and a stretched phase.

$\exists \mathbf{K} \subset \mathbb{R}^d$: convex set with non-empty interior s.t.

Phase transition

Attractive case: Transition between a collapsed phase and a stretched phase.
$\exists \mathbf{K} \subset \mathbb{R}^d$: convex set with non-empty interior s.t.

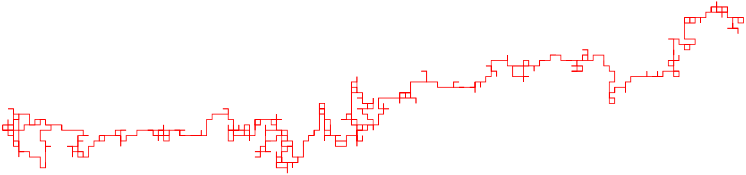


$$F \in \mathring{\mathbf{K}}$$

Phase transition

Attractive case: Transition between a collapsed phase and a stretched phase.

$\exists \mathbf{K} \subset \mathbb{R}^d$: convex set with non-empty interior s.t.



$$F \notin \mathbf{K}$$

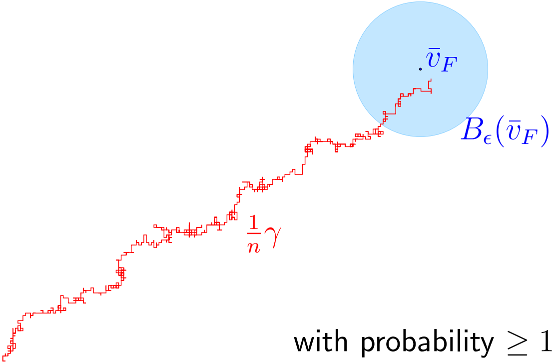
Stretched phase

The following results hold in the stretched phase, i.e. when

- $F \neq 0$ in the repulsive case
- $F \notin \mathbf{K}$ in the attractive case

Geometry of long polymers

There exists $\bar{v}_F \neq 0$ such that



with probability $\geq 1 - e^{-\kappa n}$

Geometry of long polymers

Inside $B_\epsilon(\bar{v}_F)$:

$$\mathbb{P}_n^F\Big(\frac{\gamma(n)}{n} = x\Big) = \frac{G(x)}{\sqrt{n^d}}\,\mathrm{e}^{-nJ_F(x)}\,(1+o(1))$$

G: positive and analytic on $B_\epsilon(\bar{v}_F)$
J_F: positive, analytic on $B_\epsilon(\bar{v}_F)$, and strictly convex
with a non-degenerate quadratic minimum at $\bar{v}_F$

Geometry of long polymers

Moreover, the typical shape satisfies

- **macroscale**: straight line
- **microscale**: effective random walk structure
- **mesoscale**: Brownian limit

Contents

To keep things simple, we only consider Bernoulli bond percolation on $\mathbb{Z}^d$ at $p < p_c$.

Facts: inverse correlation length

For all $x \in \mathbb{R}^d$,

$$\xi(x) = - \lim_{k\to\infty} \frac{1}{k} \log \mathbb{P}^p(0 \leftrightarrow [kx])$$

exists and is norm on $\mathbb{R}^d$. Moreover,

$$\mathbb{P}^p(0 \leftrightarrow x) \leq e^{-\xi(\vec{n}_x)|x|}$$

[Menshikov '86, Aizenman,Barsky '87]

Facts: inequalities

Harris-FKG: For all A, B increasing events,

$$\mathbb{P}^p(A \cap B) \geq \mathbb{P}^p(A)\mathbb{P}^p(B)$$

BK: For all A, B increasing events,

$$\mathbb{P}^p(A \circ B) \leq \mathbb{P}^p(A)\mathbb{P}^p(B)$$

where $\circ$ denotes disjoint occurence.

Equidecay set and Wulff shape

Two convex bodies are naturally associated to ξ:

Equidecay set and Wulff shape

Two convex bodies are naturally associated to ξ:

Equidecay set

$$\mathbf{U}_\xi = \{x \in \mathbb{R}^d \,:\, \xi(x) \leq 1\}$$

Equidecay set and Wulff shape

Two convex bodies are naturally associated to ξ:

Equidecay set

$$\mathbf{U}_\xi = \left\{x \in \mathbb{R}^d \,:\, \xi(x) \leq 1\right\}$$

Wulff shape

$$\mathbf{K}_\xi = \left\{t \in \mathbb{R}^d \,:\, (t, \vec{n})_d \leq \xi(\vec{n}),\, \forall \vec{n} \in \mathbb{S}^{d-1}\right\}$$

Equidecay set and Wulff shape

Two convex bodies are naturally associated to ξ:

Equidecay set

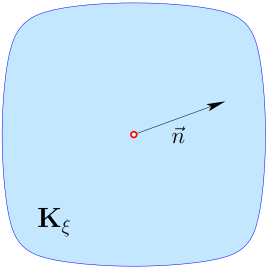
$$\mathbf{U}_\xi = \{x \in \mathbb{R}^d \, : \, \xi(x) \leq 1\}$$

Wulff shape

$$\mathbf{K}_\xi = \{t \in \mathbb{R}^d \, : \, (t, \vec{n})_d \leq \xi(\vec{n}), \, \forall \vec{n} \in \mathbb{S}^{d-1}\}$$

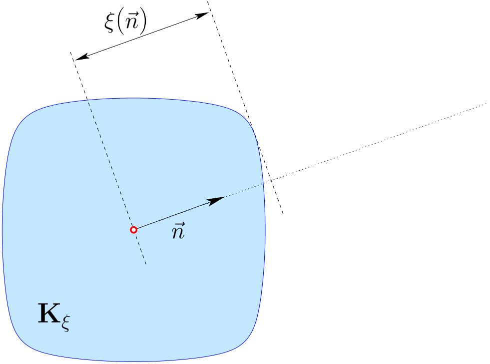
Each set encodes all the information about ξ.

Equidecay set and Wulff shape



$$\mathbf{K}_\xi = \left\{ t \in \mathbb{R}^d \, : \, (t, \vec{n})_d \leq \xi(\vec{n}), \, \forall \vec{n} \in \mathbb{S}^{d-1} \right\}$$

Equidecay set and Wulff shape



$$\mathbf{K}_\xi = \left\{ t \in \mathbb{R}^d \,:\, (t, \vec{n})_d \le \xi(\vec{n}),\ \forall \vec{n} \in \mathbb{S}^{d-1} \right\}$$

Equidecay set and Wulff shape

$\mathbf{U}_\xi$ and $\mathbf{K}_\xi$ are polar:

$$\mathbf{U}_\xi = \left\{ x \in \mathbb{R}^d \,:\, \max_{t \in \mathbf{K}_\xi} (t,x)_d \leq 1 \right\}$$

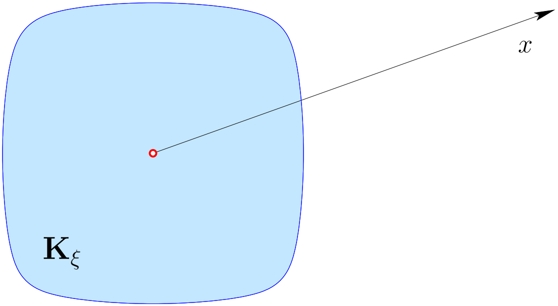
$$\mathbf{K}_\xi = \left\{ t \in \mathbb{R}^d \,:\, \max_{x \in \mathbf{U}_\xi} (t,x)_d \leq 1 \right\}$$

Equidecay set and Wulff shape

$x \in \mathbb{R}^d$ and $t \in \partial \mathbf{K}_\xi$ are dual if $(t, x)_d = \xi(x)$.

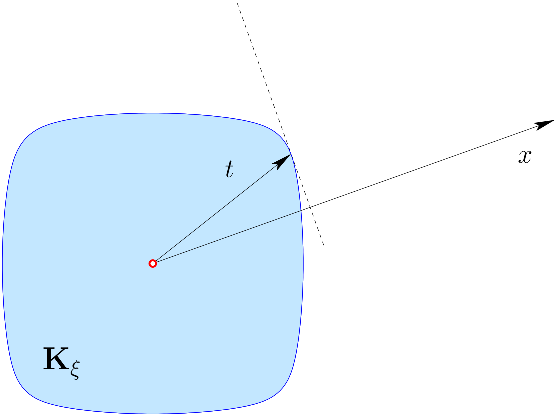
Equidecay set and Wulff shape

$x \in \mathbb{R}^d$ and $t \in \partial \mathbf{K}_\xi$ are dual if $(t, x)_d = \xi(x)$.



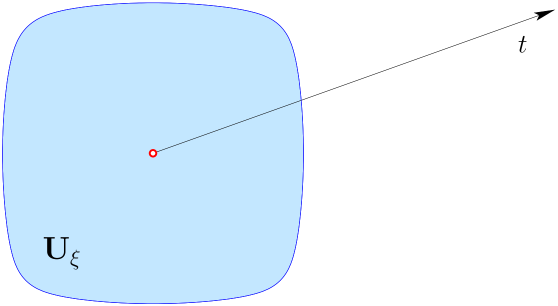
Equidecay set and Wulff shape

$x \in \mathbb{R}^d$ and $t \in \partial \mathbf{K}_\xi$ are dual if $(t, x)_d = \xi(x)$.



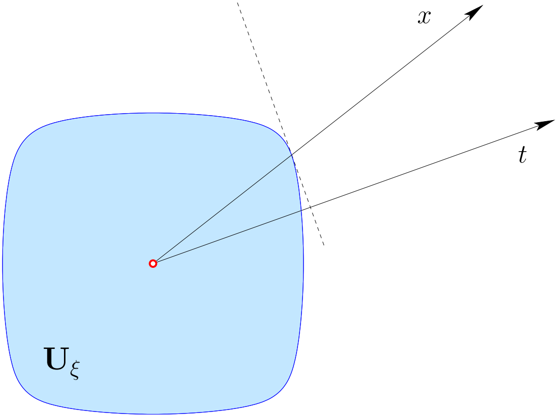
Equidecay set and Wulff shape

$x \in \mathbb{R}^d$ and $t \in \partial\mathbf{K}_\xi$ are dual if $(t, x)_d = \xi(x)$.

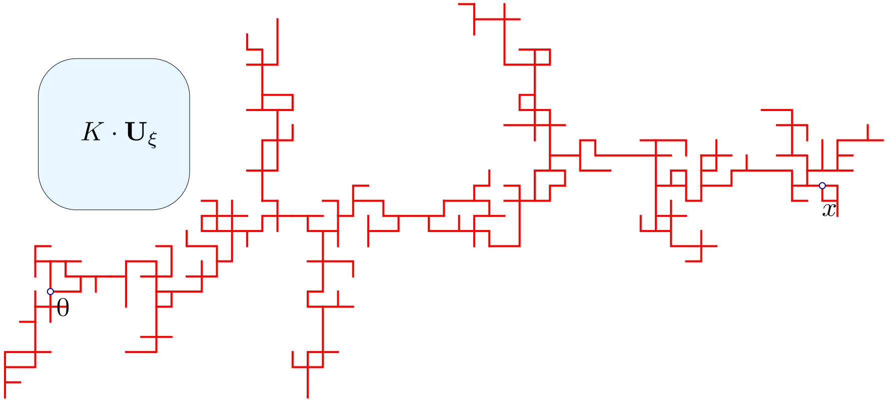


Equidecay set and Wulff shape

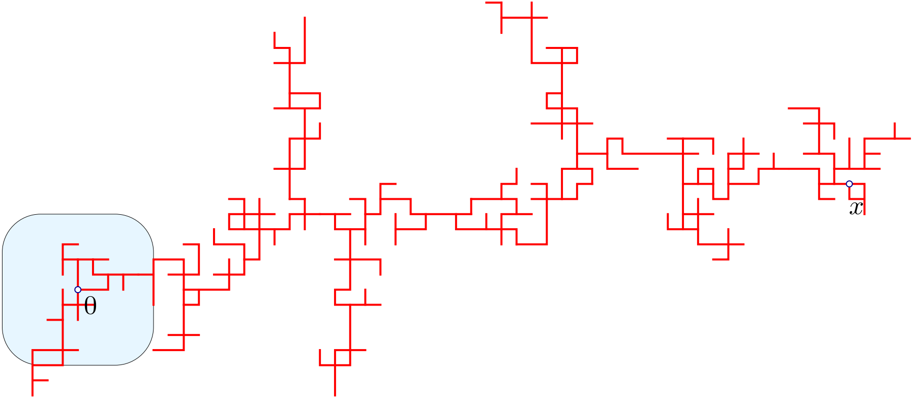
$x \in \mathbb{R}^d$ and $t \in \partial \mathbf{K}_\xi$ are dual if $(t, x)_d = \xi(x)$.



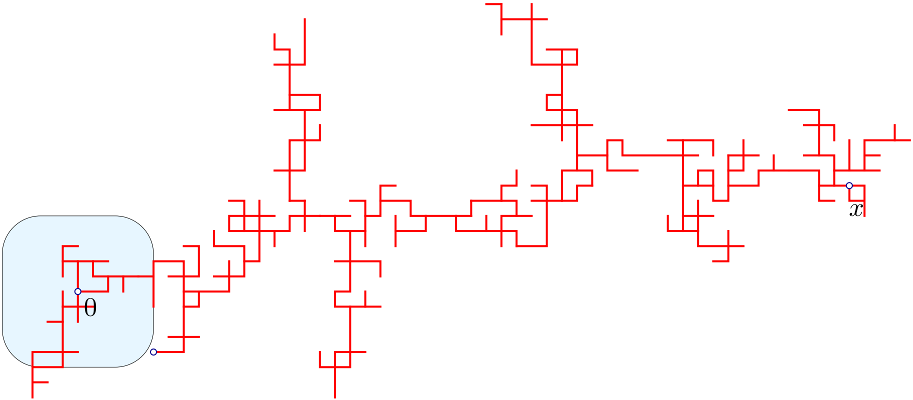
Skeleton



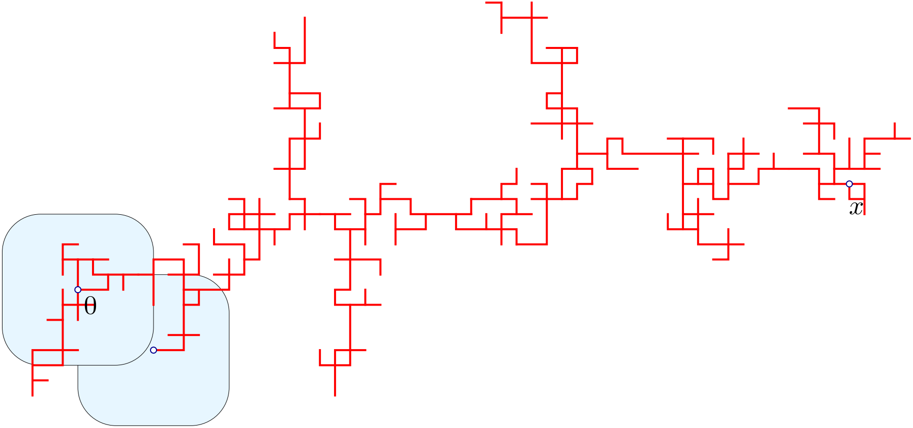
Skeleton



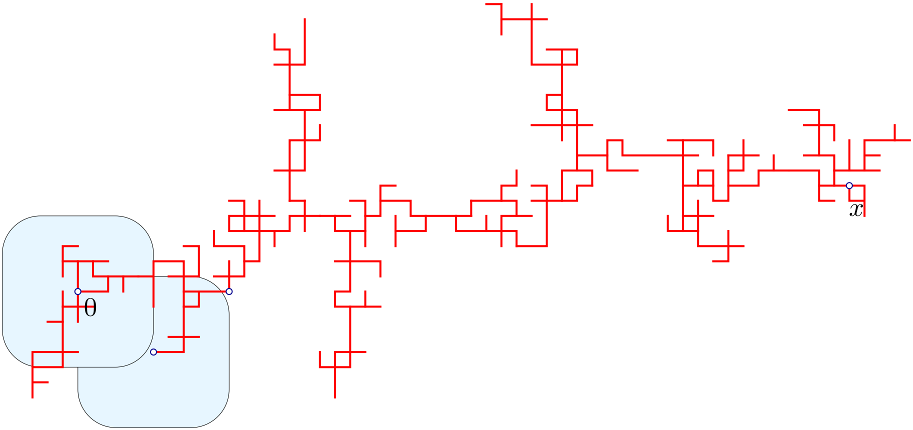
Skeleton



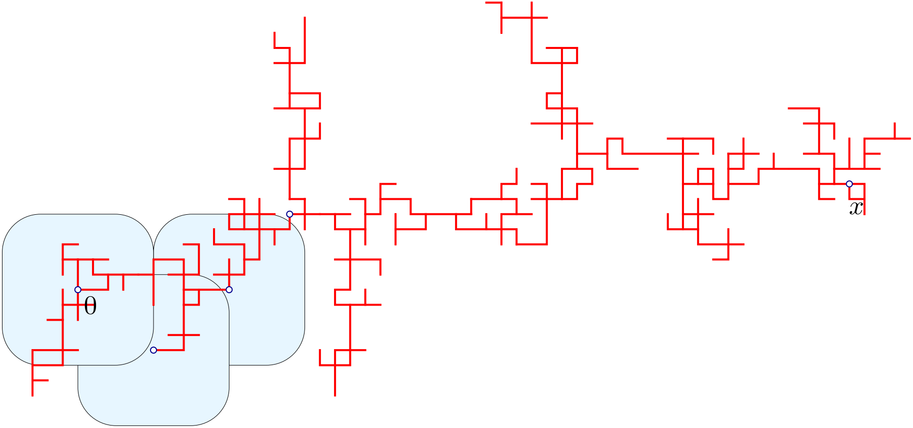
Skeleton



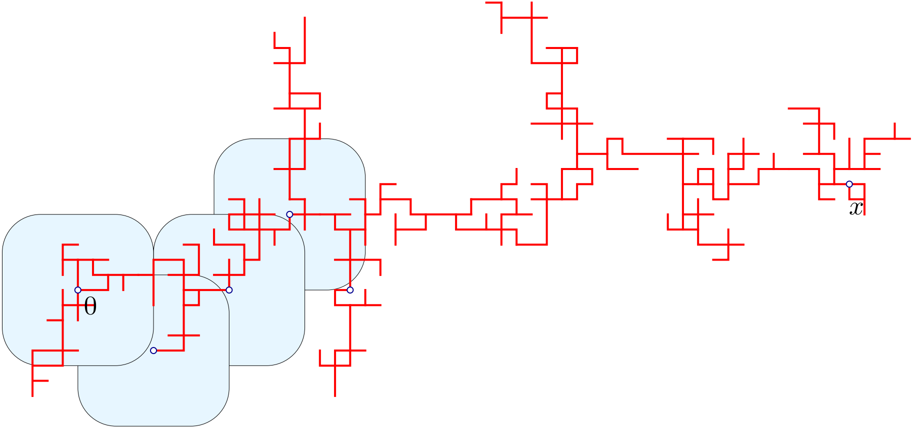
Skeleton



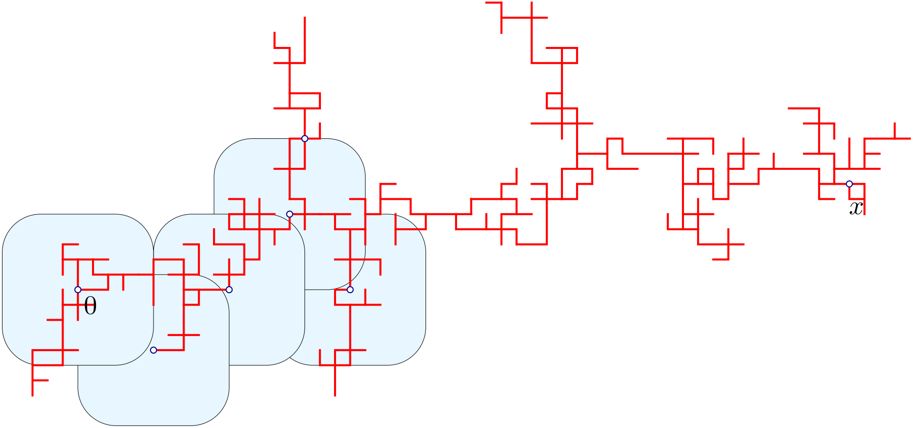
Skeleton



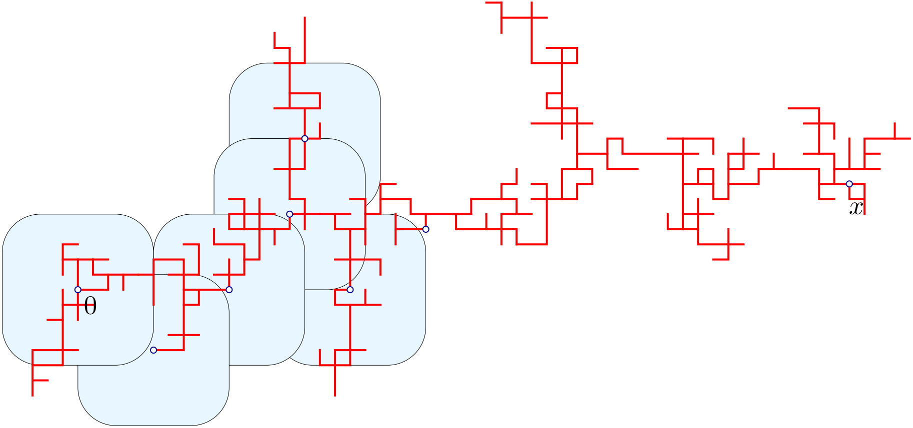
Skeleton



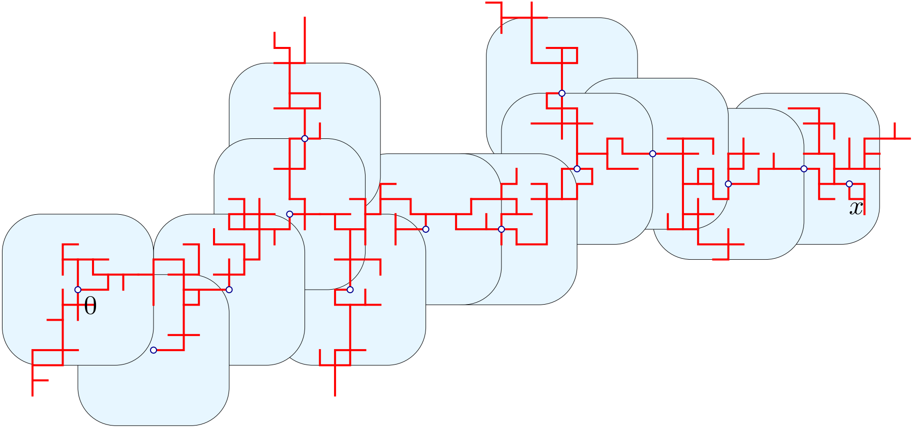
Skeleton



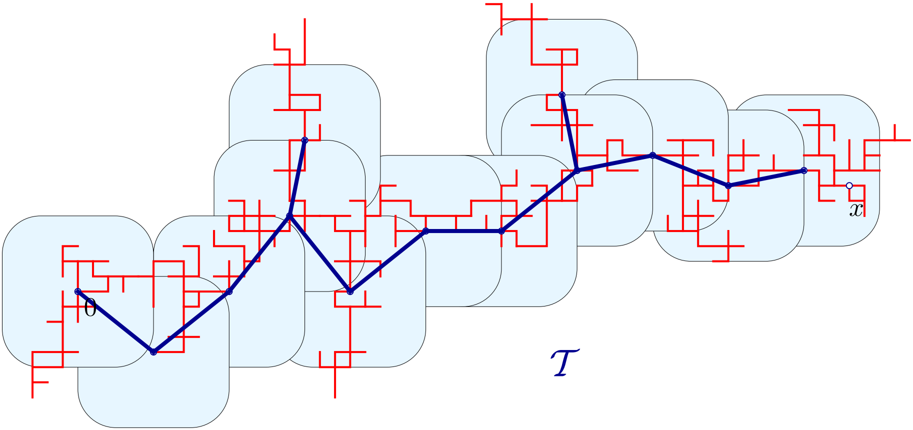
Skeleton



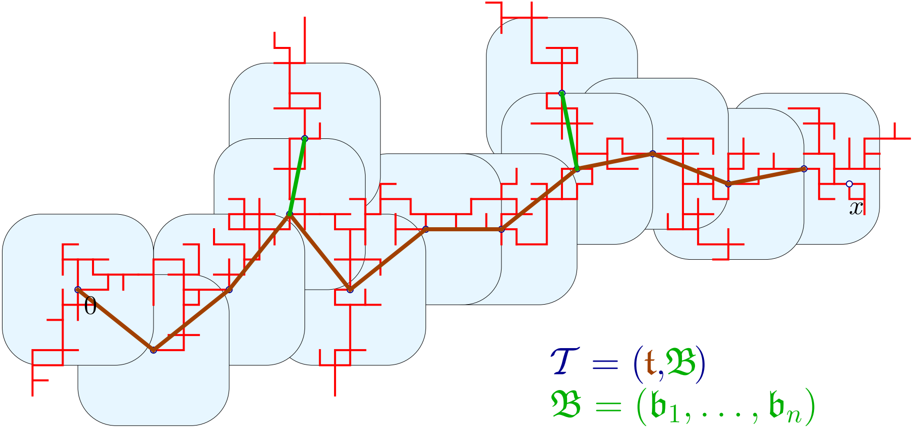
Skeleton



Skeleton

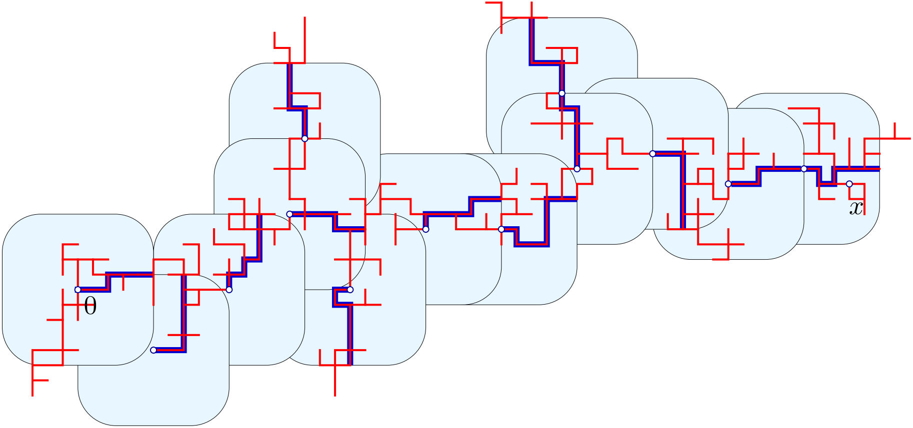


Skeleton



$$\mathcal{T} = (\mathfrak{t}, \mathfrak{B})$$
$$\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$$

Skeleton



Rough bounds

Let $\mathcal{T} = (0 = x_1, x_2, \ldots, x_{N_\mathcal{T}})$, $\mathbf{U}_\xi^K(x_i) = x_i + K\mathbf{U}_\xi$ and

$$A_i = \{x_i \leftrightarrow \partial\mathbf{U}_\xi^K(x_i)\}$$

Rough bounds

Let $\mathcal{T} = (0 = x_1, x_2, \dots, x_{N_\mathcal{T}})$, $\mathbf{U}_\xi^K(x_i) = x_i + K\mathbf{U}_\xi$ and

$$A_i = \{x_i \leftrightarrow \partial \mathbf{U}_\xi^K(x_i)\}$$

BK implies that

$$\begin{aligned}\mathbb{P}^p(\mathcal{T}) &\leq \mathbb{P}^p(A_1 \circ A_2 \circ \cdots \circ A_{N_\mathcal{T}}) \\ &\leq \prod_{i=1}^{N_\mathcal{T}} \mathbb{P}^p(A_i) = \mathbb{P}^p\big(0 \leftrightarrow \partial \mathbf{U}_\xi^K(0)\big)^{N_\mathcal{T}} \\ &\leq (cK^{d-1}e^{-K})^{N_\mathcal{T}} = e^{-KN_\mathcal{T}(1+o_K(1))}\end{aligned}$$

Rough bounds

$N_{\mathfrak{t}} = \#$ of vertices in $\mathfrak{t}$, $N_{\mathfrak{B}} = \#$ of vertices in $\mathfrak{B}$.

- Typical trees have a small trunk:

$$\exists c_1, c_2 : \mathbb{P}^p\Big(N_{\mathfrak{t}} > c_1 \frac{|x|}{K} \;\Big|\; 0 \leftrightarrow x\Big) \leq e^{-c_2|x|}$$

Rough bounds

$N_{\mathfrak{t}} = \#$ of vertices in $\mathfrak{t}$, $N_{\mathfrak{B}} = \#$ of vertices in $\mathfrak{B}$.

- Typical trees have a small trunk:

$$\exists c_1, c_2 : \; \mathbb{P}^p\big(N_{\mathfrak{t}} > c_1 \frac{|x|}{K} \;\big|\; 0 \leftrightarrow x\big) \leq e^{-c_2 |x|}$$

- Total size of branches of typical trees is small:

$$\forall c_3 > 0 : \; \mathbb{P}^p\big(N_{\mathfrak{B}} > c_3 \frac{|x|}{K} \;\big|\; 0 \leftrightarrow x\big) \leq e^{-\frac{1}{2} c_3 |x|}$$

Rough bounds

of trunks of size N $\lesssim (K^{d-1})^N = e^{N(d-1)\log K}$.

Rough bounds

of trunks of size N $\lesssim (K^{d-1})^N = e^{N(d-1)\log K}$. Therefore

$$\mathbb{P}^p(N_{\mathfrak{t}} = N) \leq e^{-KN(1-o_K(1))} e^{N(d-1)\log K} \leq e^{-KN(1-o_K(1))}.$$

Rough bounds

of trunks of size N $\lesssim (K^{d-1})^N = e^{N(d-1)\log K}$. Therefore

$$\mathbb{P}^p(N_\mathfrak{t} = N) \leq e^{-KN(1-o_K(1))} e^{N(d-1)\log K} \leq e^{-KN(1-o_K(1))}.$$

Thus (with K large enough)

$$\mathbb{P}^p(N_\mathfrak{t} \geq c_1 \frac{|x|}{K}) \leq \sum_{N \geq c_1 \frac{|x|}{K}} e^{-KN(1-o_K(1))} \leq e^{-\frac{1}{2}c_1|x|}.$$

Rough bounds

\# of trunks of size N $\lesssim (K^{d-1})^N = e^{N(d-1)\log K}$. Therefore

$$\mathbb{P}^p(N_{\mathfrak{t}} = N) \leq e^{-KN(1-o_K(1))} e^{N(d-1)\log K} \leq e^{-KN(1-o_K(1))}.$$

Thus (with K large enough)

$$\mathbb{P}^p(N_{\mathfrak{t}} \geq c_1 \frac{|x|}{K}) \leq \sum_{N \geq c_1 \frac{|x|}{K}} e^{-KN(1-o_K(1))} \leq e^{-\frac{1}{2}c_1|x|}.$$

Conclusion follows (taking c_1 large enough) since

$$\mathbb{P}^p(0 \leftrightarrow x) \geq e^{-\xi(\vec{n}_x)|x|(1+o(1))}.$$

Surcharge function

Let $t \in \partial \mathbf{K}_\xi$ be dual to x.

Surcharge function

Let $t \in \partial \mathbf{K}_\xi$ be dual to x.

The surcharge function

$$\mathfrak{s}_t(y) = \xi(y) - (t, y)_d$$

measures the typicality of an increment.

Surcharge function

Let $t \in \partial\mathbf{K}_\xi$ be dual to x.

The surcharge function

$$\mathfrak{s}_t(y) = \xi(y) - (t, y)_d$$

measures the typicality of an increment.

For a trunk $\mathfrak{t} = (\mathfrak{t}_0, \ldots, \mathfrak{t}_{N_\mathfrak{t}})$, we set

$$\mathfrak{s}_t(\mathfrak{t}) = \sum_{l=1}^{N_\mathfrak{t}} \mathfrak{s}_t(\mathfrak{t}_l - \mathfrak{t}_{l-1})$$

Surcharge function

Surcharge inequality

Let $\epsilon > 0$. There exists $K_0(\epsilon)$ such that, for all $K > K_0$,

$$\mathbb{P}\big(\mathfrak{s}_t(\mathfrak{t}) > 2\epsilon|x| \bigm| 0 \leftrightarrow x\big) \leq \mathrm{e}^{-\epsilon|x|}$$

uniformly in $x \in \mathbb{Z}^d$, $t \in \partial\mathbf{K}_\xi$ dual to x.

Surcharge function

$$\mathbb{P}^p(\mathfrak{t}) \le e^{-N_\mathfrak{t} K\,(1-o_K(1))} \le e^{-\sum_{i=1}^{N_\mathfrak{t}} \xi(\mathfrak{t}_i-\mathfrak{t}_{i-1})+o_K(1)|x|}.$$

Surcharge function

$$\mathbb{P}^p(\mathfrak{t}) \le e^{-N_\mathfrak{t} K\,(1-o_K(1))} \le e^{-\sum_{i=1}^{N_\mathfrak{t}} \xi(\mathfrak{t}_i-\mathfrak{t}_{i-1})+o_K(1)|x|}.$$

Now,

$$\begin{aligned}
\sum_{i=1}^{N_\mathfrak{t}} \xi(\mathfrak{t}_i - \mathfrak{t}_{i-1}) &= \sum_{i=1}^{N_\mathfrak{t}} \big(\mathfrak{s}_t(\mathfrak{t}_i - \mathfrak{t}_{i-1}) + (t, \mathfrak{t}_i - \mathfrak{t}_{i-1})_d\big) \\
&= \mathfrak{s}_t(\mathfrak{t}) + (t, \mathfrak{t}_{N_\mathfrak{t}})_d \\
&= \mathfrak{s}_t(\mathfrak{t}) + (t, x)_d - (t, x - \mathfrak{t}_{N_\mathfrak{t}})_d \\
&= \mathfrak{s}_t(\mathfrak{t}) + \xi(x) - (t, x - \mathfrak{t}_{N_\mathfrak{t}})_d.
\end{aligned}$$

Surcharge function

$$\mathbb{P}^p(\mathfrak{t}) \le e^{-N_\mathfrak{t} K\,(1-o_K(1))} \le e^{-\sum_{i=1}^{N_\mathfrak{t}} \xi(\mathfrak{t}_i-\mathfrak{t}_{i-1})+o_K(1)|x|}.$$

Now,

$$\begin{aligned}\sum_{i=1}^{N_\mathfrak{t}} \xi(\mathfrak{t}_i-\mathfrak{t}_{i-1}) &= \sum_{i=1}^{N_\mathfrak{t}} \bigl(\mathfrak{s}_t(\mathfrak{t}_i-\mathfrak{t}_{i-1}) + (t,\mathfrak{t}_i-\mathfrak{t}_{i-1})_d\bigr)\\ &= \mathfrak{s}_t(\mathfrak{t}) + (t,\mathfrak{t}_{N_\mathfrak{t}})_d\\ &= \mathfrak{s}_t(\mathfrak{t}) + (t,x)_d - (t,x-\mathfrak{t}_{N_\mathfrak{t}})_d\\ &= \mathfrak{s}_t(\mathfrak{t}) + \xi(x) - (t,x-\mathfrak{t}_{N_\mathfrak{t}})_d.\end{aligned}$$

$$\Longrightarrow\ \mathbb{P}^p(\mathfrak{t}) \le e^{-\mathfrak{s}_t(\mathfrak{t})-\xi(x)+o_K(1)|x|}.$$

Surcharge function

We can assume that $N_{\mathfrak{t}} \leq c_1 |x|/K$.

Surcharge function

We can assume that $N_{\mathfrak{t}} \leq c_1|x|/K$. The number of such trunks is bounded above by

$$e^{c(\log K/K)|x|} = e^{o_K(1)|x|}.$$

Surcharge function

We can assume that $N_{\mathfrak{t}} \leq c_1|x|/K$. The number of such trunks is bounded above by

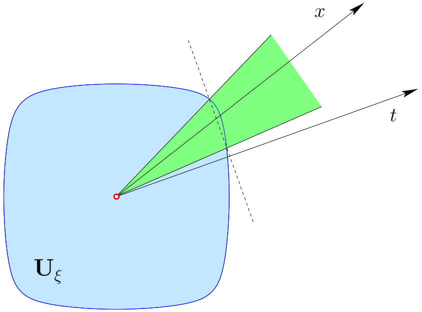
$$e^{c(\log K/K)|x|} = e^{o_K(1)|x|}.$$

Since, $\mathbb{P}^p(\mathfrak{t}) \leq e^{-\mathfrak{s}_t(\mathfrak{t})-\xi(x)+o_K(1)|x|}$,

$$\mathbb{P}^p(\mathfrak{s}_t(\mathfrak{t}) \geq 2\epsilon|x|) \leq e^{-(2\epsilon-o_K(1))|x|}e^{-\xi(x)},$$

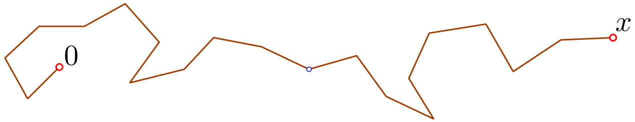
and the conclusion follows as before... □

Forward cone

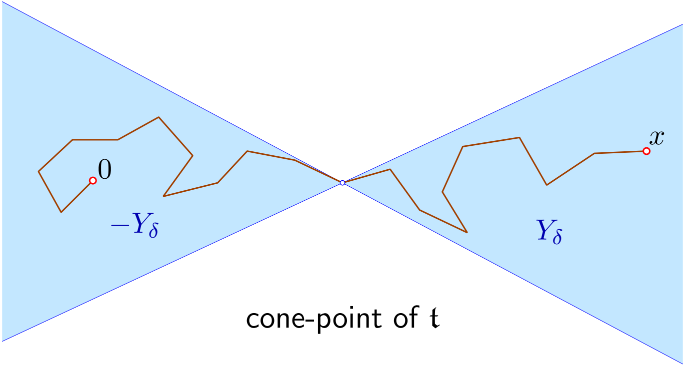


$$Y_\delta(t) = \{y \in \mathbb{R}^d : (y,t)_d > (1-\delta)\xi(y)\}$$
$$= \{y \in \mathbb{R}^d : \mathfrak{s}_t(y) < \delta\xi(y)\}$$

Cone points of trunks



Cone points of trunks



cone-point of $\mathfrak{t}$

Cone points of trunks

$\#^{\text{n.c.p.}}(\mathfrak{t}) = \#\{\text{non-cone-points of } \mathfrak{t}\}$

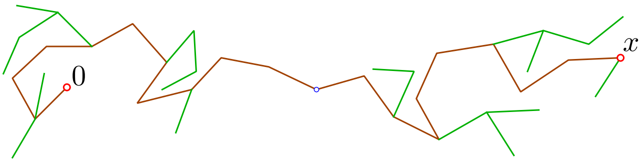
Lemma

$$\mathfrak{s}_t(\mathfrak{t}) \geq c_4\, \delta\, K\, \#^{\text{n.c.p.}}(\mathfrak{t})$$

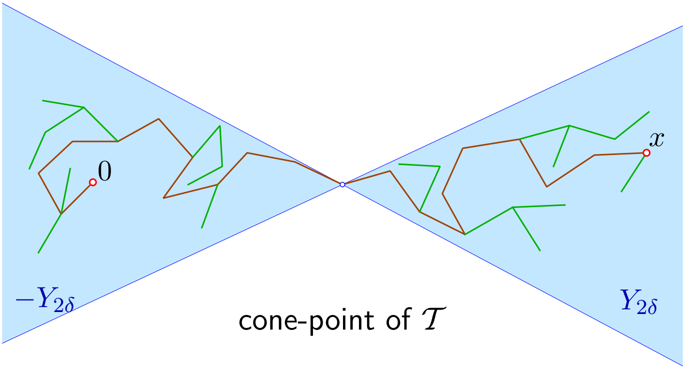
Consequently,

$$\mathbb{P}\big(\#^{\text{n.c.p.}}(\mathfrak{t}) \geq \epsilon N(\mathfrak{t}) \;\big|\; 0 \leftrightarrow x\big) \leq \mathrm{e}^{-c_5 \epsilon |x|}$$

Cone points of trees



Cone points of trees

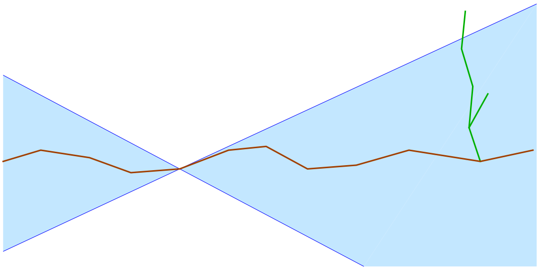


Cone points of trees

- A cone-point of $\mathfrak{t}$ but not of $\mathcal{T}$ is called blocked.

Cone points of trees

- A cone-point of $\mathfrak{t}$ but not of $\mathcal{T}$ is called blocked.
- The number of cone-points of $\mathfrak{t}$ that a branch can block is proportional to its size.



Cone points of trees

- A cone-point of $\mathfrak{t}$ but not of $\mathcal{T}$ is called blocked.
- The number of cone-points of $\mathfrak{t}$ that a branch can block is proportional to its size.
- The total size of the branches is small $\implies$ only few cone-points of trunk can be blocked.

Cone points of trees

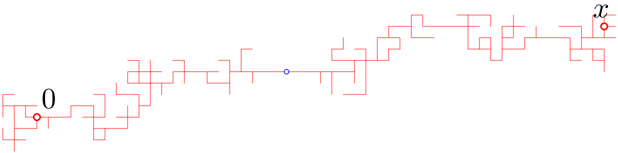
- A cone-point of $\mathfrak{t}$ but not of $\mathcal{T}$ is called blocked.
- The number of cone-points of $\mathfrak{t}$ that a branch can block is proportional to its size.
- The total size of the branches is small $\implies$ only few cone-points of trunk can be blocked.

Lemma

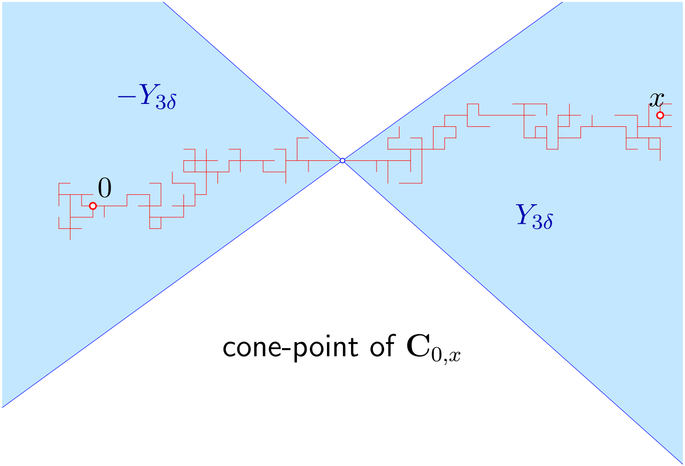
There exist $\nu > 0$ and c such that

$$\mathbb{P}\big(\#\{\text{cone-points of } \mathcal{T}\} < \nu \frac{|x|}{K} \mid 0 \leftrightarrow x\big) \leq \mathrm{e}^{-c|x|}$$

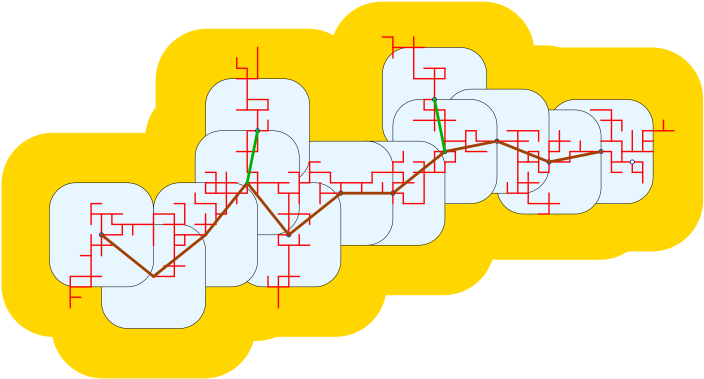
Cone points of clusters



Cone points of clusters

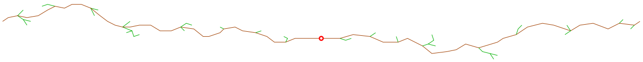


Cone points of clusters

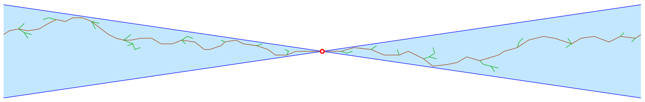


Clusters remain close to their approximating tree

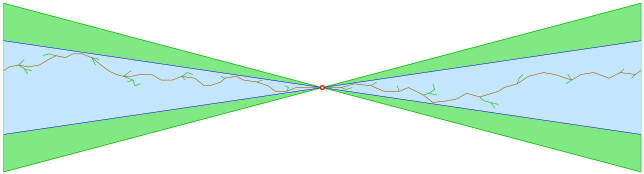
Cone points of clusters



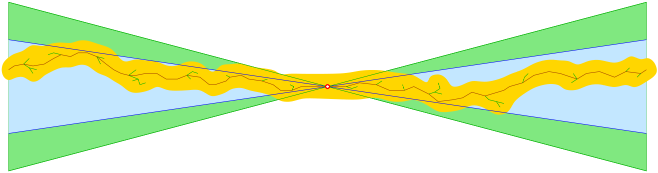
Cone points of clusters



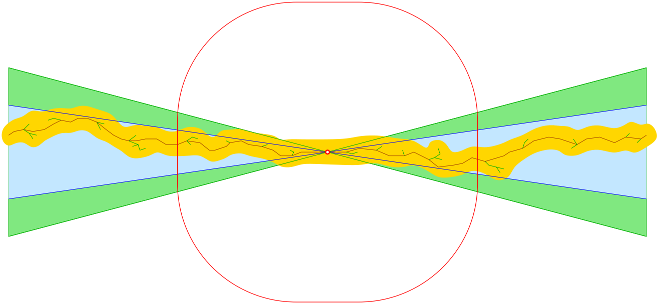
Cone points of clusters



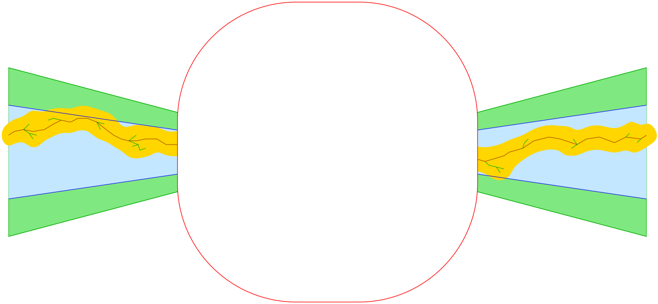
Cone points of clusters



Cone points of clusters



Cone points of clusters



Cone points of clusters

- Outside this finite ball, the cone condition is automatically satisfied.

Cone points of clusters

- Outside this finite ball, the cone condition is automatically satisfied.
- Inside this finite ball, there is a strictly positive probability that the cluster remains inside the cone, uniformly in what happens elsewhere.

Cone points of clusters

- Outside this finite ball, the cone condition is automatically satisfied.
- Inside this finite ball, there is a strictly positive probability that the cluster remains inside the cone, uniformly in what happens elsewhere.

$$\Downarrow$$

- Up to exponentially small error, a positive density of the cone-points of $\mathcal{T}$ are also cone-points of $\mathbf{C}_{0,x}$

Cone points of clusters

$\#^{\text{cone}}_{t,\delta}(\mathbf{C}_{0,x})$: number of cone-points of $\mathbf{C}_{0,x}$

Theorem

There exist $\delta \in (0, \frac{1}{2})$, ν and c such that

$$\mathbb{P}\Big(\#^{\text{cone}}_{t,\delta}(\mathbf{C}_{0,x}) \leq \nu|x| \;\Big|\; 0 \leftrightarrow x\Big) \leq \mathrm{e}^{-c|x|}$$

uniformly in x and dual t.

Decomposition into irreducible pieces

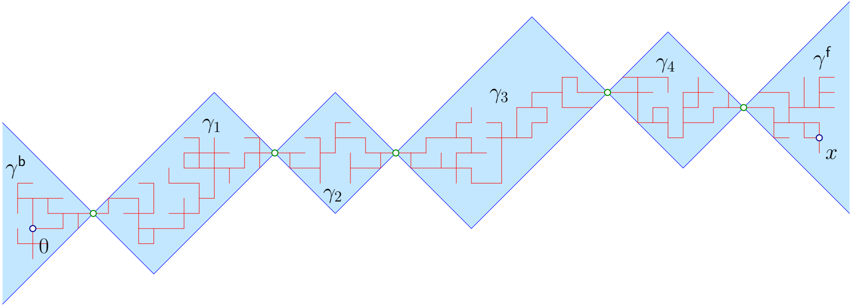
We can thus decompose the cluster $\mathbf{C}_{0,x}$ into irreducible pieces:

$$\mathbf{C}_{0,x} = \gamma^{\mathsf{b}} \amalg \gamma_1 \amalg \ldots \amalg \gamma_n \amalg \gamma^{\mathsf{f}}$$

Decomposition into irreducible pieces

We can thus decompose the cluster $\mathbf{C}_{0,x}$ into irreducible pieces:

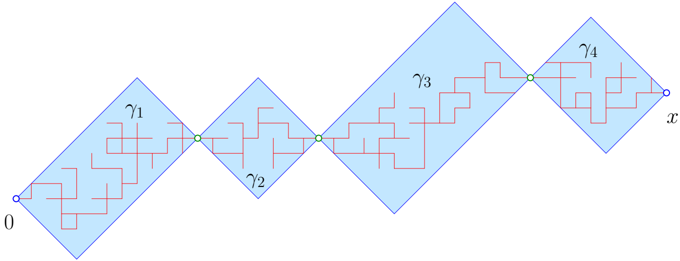
$$\mathbf{C}_{0,x} = \gamma^{\mathsf{b}} \amalg \gamma_1 \amalg \ldots \amalg \gamma_n \amalg \gamma^{\mathsf{f}}$$



Decomposition into irreducible pieces

To simplify, I shall assume in the sequel that:

$$\mathbf{C}_{0,x} = \gamma_1 \amalg \ldots \amalg \gamma_n$$



Decomposition into irreducible pieces

We can thus write

$$\mathbb{P}(0 \leftrightarrow x) \approx \sum_{n\geq 1} \sum_{\substack{\gamma_1,\ldots,\gamma_n \\ \sum D(\gamma_i)=x \\ \text{irred.}}} \mathbb{P}(C_0 = \gamma_1 \amalg \ldots \amalg \gamma_n)$$

For $\gamma : y \to z$, $D(\gamma) = z - y$.

Decomposition into irreducible pieces

We can thus write

$$e^{\xi(x)}\,\mathbb{P}(0 \leftrightarrow x) \approx \sum_{n\geq 1} \sum_{\substack{\gamma_1,\dots,\gamma_n \\ \sum D(\gamma_i)=x \\ \text{irred.}}} \mathbb{P}(C_0 = \gamma_1 \amalg \dots \amalg \gamma_n)\, e^{\xi(x)}$$

For $\gamma : y \to z$, $D(\gamma) = z - y$.

Decomposition into irreducible pieces

We can thus write

$$e^{\xi(x)}\,\mathbb{P}(0 \leftrightarrow x) \approx \sum_{n\geq 1} \sum_{\substack{\gamma_1,\dots,\gamma_n \\ \sum D(\gamma_i)=x \\ \text{irred.}}} \mathbb{P}(C_0 = \gamma_1 \amalg \ldots \amalg \gamma_n)\, e^{\xi(x)}$$

$$= \sum_{n\geq 1} \sum_{\substack{\gamma_1,\dots,\gamma_n \\ \sum D(\gamma_i)=x \\ \text{irred.}}} \mathbb{P}(C_0 = \gamma_1 \amalg \ldots \amalg \gamma_n)\, e^{(t,x)_d}$$

For $\gamma : y \to z$, $D(\gamma) = z - y$.

Decomposition into irreducible pieces

We can thus write

$$e^{\xi(x)}\,\mathbb{P}(0\leftrightarrow x)\approx\sum_{n\geq 1}\sum_{\substack{\gamma_1,\dots,\gamma_n\\ \sum D(\gamma_i)=x\\ \text{irred.}}}\mathbb{P}(C_0=\gamma_1\amalg\dots\amalg\gamma_n)\,e^{\xi(x)}$$

$$=\sum_{n\geq 1}\sum_{\substack{\gamma_1,\dots,\gamma_n\\ \sum D(\gamma_i)=x\\ \text{irred.}}}\mathbb{P}(C_0=\gamma_1\amalg\dots\amalg\gamma_n)\,e^{(t,x)_d}$$

$$=\sum_{n\geq 1}\sum_{\substack{\gamma_1,\dots,\gamma_n\\ \sum D(\gamma_i)=x\\ \text{irred.}}}\mathbb{P}(C_0=\gamma_1\amalg\dots\amalg\gamma_n)\prod_{i=1}^{n}e^{(t,D(\gamma_i))_d}$$

For $\gamma : y\to z$, $D(\gamma)=z-y$.

Decomposition into irreducible pieces

Thanks to independence of edge states in Bernoulli percolation,

$$\mathbb{P}(C_0 = \gamma_1 \amalg \ldots \amalg \gamma_n) = \prod_{i=1}^{n} w(\gamma_i),$$

where w is morally given by $w(\gamma) = p^{|\gamma|}(1-p)^{|\partial\gamma|}$.

Decomposition into irreducible pieces

We set, for $x \in \mathbb{Z}^d$,

$$\mathbb{Q}(x) = e^{(t,x)_d} \sum_{\substack{\gamma:\, 0 \to x \\ \text{irred.}}} w(\gamma).$$

We then have

- $\mathbb{Q}$ is a probability measure on $\mathbb{Z}^d$;
- $\mathbb{Q}(|x| > \ell) \leq e^{-c\ell}$, for some $c > 0$.

Decomposition into irreducible pieces

We can thus write

$$e^{\xi(x)}\,\mathbb{P}(0 \leftrightarrow x) \approx \sum_{n\geq 1} \sum_{\substack{\gamma_1,\dots,\gamma_n \\ \sum D(\gamma_i)=x \\ \text{irred.}}} \mathbb{P}(C_0 = \gamma_1 \amalg \dots \amalg \gamma_n) \prod_{i=1}^{n} e^{(t,D(\gamma_i))_d}$$

Decomposition into irreducible pieces

We can thus write

$$e^{\xi(x)}\,\mathbb{P}(0\leftrightarrow x) \approx \sum_{n\geq 1}\sum_{\substack{\gamma_1,\dots,\gamma_n\\ \sum D(\gamma_i)=x\\ \text{irred.}}} \mathbb{P}(C_0=\gamma_1\amalg\ldots\amalg\gamma_n)\prod_{i=1}^{n} e^{(t,D(\gamma_i))_d}$$

$$=\sum_{n\geq 1}\sum_{\substack{x_1,\dots,x_n\\ \sum x_i=x}}\prod_{i=1}^{n}\mathbb{Q}(x_i)$$

Decomposition into irreducible pieces

We can thus write

$$e^{\xi(x)}\,\mathbb{P}(0\leftrightarrow x) \approx \sum_{n\geq 1}\sum_{\substack{\gamma_1,\dots,\gamma_n\\ \sum D(\gamma_i)=x\\ \text{irred.}}} \mathbb{P}(C_0=\gamma_1 \amalg \dots \amalg \gamma_n)\prod_{i=1}^{n} e^{(t,D(\gamma_i))_d}$$

$$= \sum_{n\geq 1}\sum_{\substack{x_1,\dots,x_n\\ \sum x_i = x}}\prod_{i=1}^{n}\mathbb{Q}(x_i)$$

$$= \text{Prob}(\exists n\geq 1\,:\,X_n=x),$$

where X is a (directed) random walk on $\mathbb{Z}^d$ with i.i.d. increments of law $\mathbb{Q}$.

Decomposition into irreducible pieces

Ornstein-Zernike asymptotics now easily follow from the local limit theorem for i.i.d. random variables with small exponential moments:

$$e^{\xi(x)}\,\mathbb{P}(0 \leftrightarrow x) \approx \text{Prob}(\exists n \geq 1 \,:\, X_n = x) = \frac{C(t)}{|x|^{(d-1)/2}}(1 + o(1)),$$

where t being dual to x only depends on $x/|x|$.

"Proof by analogy" that $\mathbb{Q}$ is a probability measure on $\mathbb{Z}^d$:

$$0 \le g_n \longleftrightarrow e^{\xi(x)}\mathbb{P}^p(0 \leftrightarrow x),$$
$$0 \le f_n \longleftrightarrow e^{\xi(x)}\mathbb{P}^p(0 \stackrel{\text{irr.}}{\leftrightarrow} x) = \mathbb{Q}(x)$$

"Proof by analogy" that $\mathbb{Q}$ is a probability measure on $\mathbb{Z}^d$:

$$0 \le g_n \longleftrightarrow e^{\xi(x)} \mathbb{P}^p(0 \leftrightarrow x),$$
$$0 \le f_n \longleftrightarrow e^{\xi(x)} \mathbb{P}^p(0 \overset{\text{irr.}}{\leftrightarrow} x) = \mathbb{Q}(x)$$

Renewal relation: $g_0 = 1$, $g_n = \sum_{k=1}^{n} f_k g_{n-k}$.

"Proof by analogy" that $\mathbb{Q}$ is a probability measure on $\mathbb{Z}^d$:

$$0 \le g_n \longleftrightarrow e^{\xi(x)}\mathbb{P}^p(0 \leftrightarrow x),$$

$$0 \le f_n \longleftrightarrow e^{\xi(x)}\mathbb{P}^p(0 \stackrel{\text{irr.}}{\leftrightarrow} x) = \mathbb{Q}(x)$$

Renewal relation: $g_0 = 1$, $g_n = \sum_{k=1}^{n} f_k g_{n-k}$.

Generating functions: $\mathbb{G}(z) = \sum_{n\ge 0} g_n z^n$, $\mathbb{F}(z) = \sum_{n\ge 1} f_n z^n$.

"Proof by analogy" that $\mathbb{Q}$ is a probability measure on $\mathbb{Z}^d$:

$$0 \le g_n \longleftrightarrow e^{\xi(x)}\mathbb{P}^p(0 \leftrightarrow x),$$
$$0 \le f_n \longleftrightarrow e^{\xi(x)}\mathbb{P}^p(0 \overset{\text{irr.}}{\leftrightarrow} x) = \mathbb{Q}(x)$$

Renewal relation: $g_0 = 1$, $g_n = \sum_{k=1}^{n} f_k g_{n-k}$.

Generating functions: $\mathbb{G}(z) = \sum_{n\ge 0} g_n z^n$, $\mathbb{F}(z) = \sum_{n\ge 1} f_n z^n$.

By assumption: $\mathbb{G}$ has radius of conv. 1.

Separation of masses: $\mathbb{F}$ has radius of conv. > 1.

"Proof by analogy" that $\mathbb{Q}$ is a probability measure on $\mathbb{Z}^d$:

$$0 \le g_n \longleftrightarrow e^{\xi(x)}\mathbb{P}^p(0 \leftrightarrow x),$$
$$0 \le f_n \longleftrightarrow e^{\xi(x)}\mathbb{P}^p(0 \overset{\text{irr.}}{\leftrightarrow} x) = \mathbb{Q}(x)$$

Renewal relation: $g_0 = 1$, $g_n = \sum_{k=1}^n f_k g_{n-k}$.

Generating functions: $\mathbb{G}(z) = \sum_{n\ge0} g_n z^n$, $\mathbb{F}(z) = \sum_{n\ge1} f_n z^n$.

By assumption: $\mathbb{G}$ has radius of conv. 1.

Separation of masses: $\mathbb{F}$ has radius of conv. > 1.

Renewal $\implies$ $\mathbb{G}(z) = 1 + \mathbb{F}(z)\mathbb{G}(z)$, $\mathbb{G}(z) = (1 - \mathbb{F}(z))^{-1}$.

"Proof by analogy" that $\mathbb{Q}$ is a probability measure on $\mathbb{Z}^d$:

$$0 \le g_n \longleftrightarrow e^{\xi(x)} \mathbb{P}^p(0 \leftrightarrow x),$$
$$0 \le f_n \longleftrightarrow e^{\xi(x)} \mathbb{P}^p(0 \overset{\text{irr.}}{\leftrightarrow} x) = \mathbb{Q}(x)$$

Renewal relation: $g_0 = 1$, $g_n = \sum_{k=1}^{n} f_k g_{n-k}$.

Generating functions: $\mathbb{G}(z) = \sum_{n\ge 0} g_n z^n$, $\mathbb{F}(z) = \sum_{n\ge 1} f_n z^n$.

By assumption: $\mathbb{G}$ has radius of conv. 1.

Separation of masses: $\mathbb{F}$ has radius of conv. > 1.

Renewal $\implies$ $\mathbb{G}(z) = 1 + \mathbb{F}(z)\mathbb{G}(z)$, $\mathbb{G}(z) = (1 - \mathbb{F}(z))^{-1}$.

$\implies$ $\mathbb{F}(1) = 1$

which is equivalent to $\sum_{k\ge 1} f_k = 1$.

More general situations

Some problems with the above argument in more general cases (say, FK percolation with $q > 1$):

- No BK inequality $\implies$ we don't get upper bounds on skeletons weights for free anymore.

More general situations

Some problems with the above argument in more general cases (say, FK percolation with $q > 1$):

- No BK inequality $\implies$ we don't get upper bounds on skeletons weights for free anymore.
- The increments of the effective random walk are not independent anymore $\implies$ we cannot rely on the local limit theorem for i.i.d. random variables anymore.

More general situations

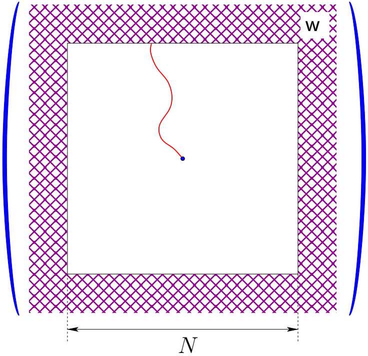
Some problems with the above argument in more general cases (say, FK percolation with $q > 1$):

- No BK inequality $\implies$ we don't get upper bounds on skeletons weights for free anymore.
- The increments of the effective random walk are not independent anymore $\implies$ we cannot rely on the local limit theorem for i.i.d. random variables anymore.

Both can be dealt with using suitable exponential mixing properties (and extending the local limit theorem from i.i.d. to random variables with exponential mixing).

Absence of BK: FK-percolation with $q > 1$

We *assume* that p is such that there exist $\nu_0, \nu_1 > 0$ s.t., $\forall N$,

$\mathbb{P}^{p,q}$  $\leq \nu_0 e^{-\nu_1 N}$

Absence of BK: FK-percolation with $q > 1$

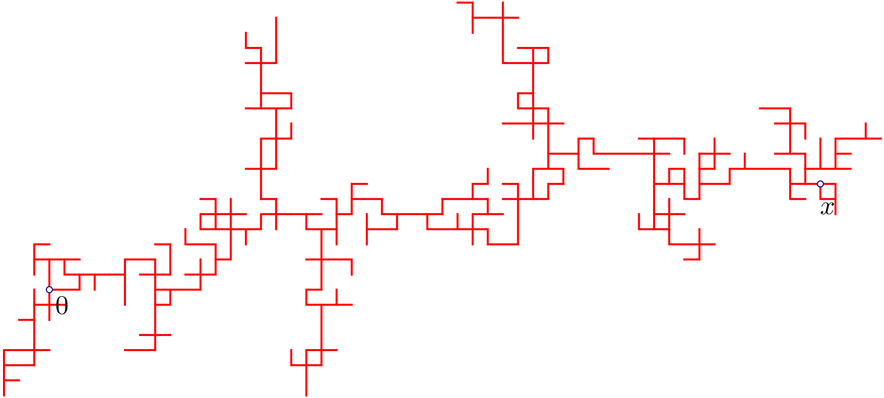
Conjecture

This is true for all $p < p_c(q)$.

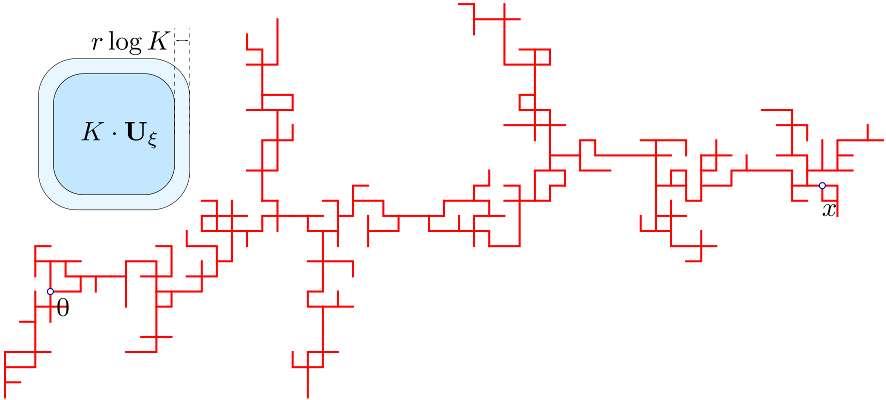
Known $(\forall d)$ when:

- $q = 1$ [Aizenman-Barsky '87]
- $q = 2$ [Aizenman *et al* '87]
- $q \gg 1$ [Laanait *et al* '91]

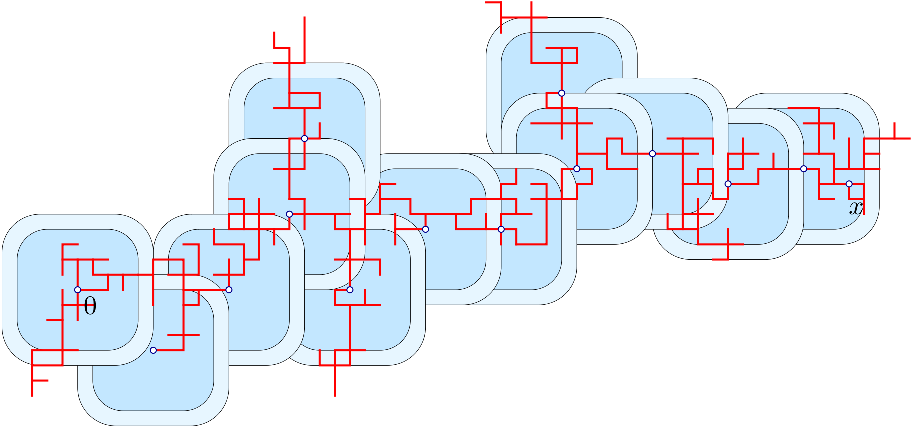
Absence of BK: FK-percolation with $q > 1$



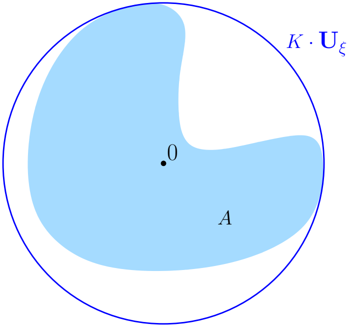
Absence of BK: FK-percolation with $q > 1$



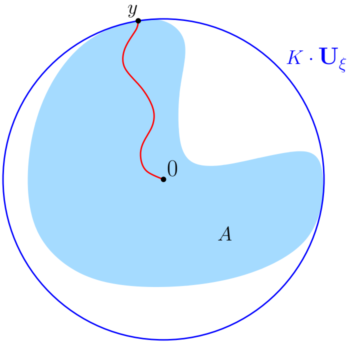
Absence of BK: FK-percolation with $q > 1$



Mixing for connectivities

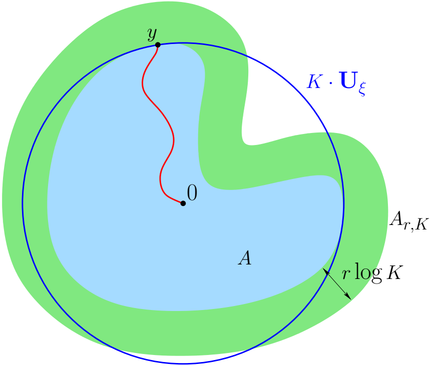


Mixing for connectivities



$\mathbb{P}^{p,q}(0 \overset{A}{\leftrightarrow} y \qquad)$

Mixing for connectivities



$$\sup_{\bar\omega} \mathbb{P}^{p,q}(0 \overset{A}{\leftrightarrow} y \,|\, \omega \equiv \bar\omega \text{ off } A_{r,K}) \leq e^{-K}\,(1 + o_K(1))$$

Limit theorem

For $q > 1$, $e^{\xi(x)}\mathbb{P}(C_0 = \gamma_1 \amalg \ldots \amalg \gamma_n)$ does not factorize anymore. However, we can write

$$e^{\xi(x)}\mathbb{P}(C_0 = \gamma_1 \amalg \cdots \amalg \gamma_n) \\ = \mathbb{Q}(\gamma_1)\mathbb{Q}(\gamma_2|\gamma_1)\cdots\mathbb{Q}(\gamma_n|\gamma_1 \amalg \gamma_2 \amalg \cdots \amalg \gamma_{n-1}),$$

for some suitable measure $\mathbb{Q}$ on finite strings of irreducible paths.

Limit theorem

For $q > 1$, $e^{\xi(x)}\mathbb{P}(C_0 = \gamma_1 \amalg \ldots \amalg \gamma_n)$ does not factorize anymore. However, we can write

$$e^{\xi(x)}\mathbb{P}(C_0 = \gamma_1 \amalg \cdots \amalg \gamma_n) = \mathbb{Q}(\gamma_1)\mathbb{Q}(\gamma_2|\gamma_1)\cdots\mathbb{Q}(\gamma_n|\gamma_1 \amalg \gamma_2 \amalg \cdots \amalg \gamma_{n-1}),$$

for some suitable measure $\mathbb{Q}$ on finite strings of irreducible paths. Moreover,

$$\frac{\mathbb{Q}(\gamma_k|\gamma_1 \amalg \cdots \amalg \gamma_\ell \amalg \gamma_{\ell+1} \amalg \cdots \amalg \gamma_{k-1})}{\mathbb{Q}(\gamma_k|\widetilde{\gamma}_1 \amalg \cdots \amalg \widetilde{\gamma}_\ell \amalg \gamma_{\ell+1} \amalg \cdots \amalg \gamma_{k-1})} \leq e^{-c(k-l)}.$$

Under these conditions, it is possible to extend the local limit theorem.